Fundamental Solutions for the Klein-Gordon Equation in de Sitter Spacetime

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Abstract

In this article we construct the fundamental solutions for the Klein-Gordon equation in de Sitter spacetime. We use these fundamental solutions to represent solutions of the Cauchy problem and to prove $L^p - L^q$ estimates for the solutions of the equation with and without a source term.

0 Introduction and Statement of Results

In this paper we construct the fundamental solutions for the Klein-Gordon equation in the de Sitter spacetime and use these fundamental solutions to find representations of the solutions to the Cauchy problem as well as $L^p - L^q$ estimates for them.

After averaging on a suitable scale, our universe is homogeneous and isotropic; therefore, the properties of the universe can be properly described by treating the matter as a perfect homogeneous fluid. In the models of the universe proposed by Einstein [12] and de Sitter [11] the line element is connected with the proper mass density and the proper pressure in the universe by the field equations for a perfect fluid. There are two alternatives which lead to the solutions of Einstein and de Sitter, respectively [22, Sec.132].

In the models proposed by Einstein [12] and de Sitter [11] the universe is assumed to be a *static* system, which means that we can introduce a system of coordinates $x^i = (r, \theta, \phi, ct)$ in which the line element has the static and spherically symmetric form

$$ds^2 = -b(r)c^2dt^2 + a(r)dr^2 + r^2(d\theta^2 + \sin^2 d\phi^2),$$

where a and b are functions of r only. On account of the assumed homogeneity of the universe any point in the space may be taken as the origin $r = 0$ of the spatial system of coordinates. The functions $a(r)$ and $b(r)$ are connected with the proper mass density μ and the proper pressure p in the universe by the field equations for a perfect fluid

$$(\mu c^2 + p)b' = 0, \tag{0.1}$$

$$\frac{b'}{abr} - \frac{1}{r^2}\left(1 - \frac{1}{a}\right) + \Lambda = \kappa p, \tag{0.2}$$

$$\frac{a'}{a^2 r} + \frac{1}{r^2}\left(1 - \frac{1}{a}\right) - \Lambda = \kappa\mu c^2, \tag{0.3}$$

where Λ is *cosmological constant*, while p and μ are constants. The general solution of the equation (0.3) is

$$a(r) = \left(1 - \frac{2M_{bh}}{r} - \frac{\Lambda r^2}{3}\right)^{-1}. \tag{0.4}$$

The constant of integration M_{bh} may have a meaning of the "mass of the black holes". There are two alternatives $b' = 0$ or $\mu c^2 + p = 0$, which lead to the solutions of Einstein and de Sitter, respectively. In the case of de Sitter universe $ab = constant$. By a trivial change of scale of the time variable, this constant can, of course, always be made equal to 1, which means

$$b(r) = 1 - \frac{2M_{bh}}{r} - \frac{\Lambda r^2}{3}\,.$$

The corresponding metric with the line element

$$ds^2 = -\left(1 - \frac{2M_{bh}}{r} - \frac{\Lambda r^2}{3}\right) c^2\, dt^2 + \left(1 - \frac{2M_{bh}}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$$

is called the Schwarzschild - de Sitter metric. The Cauchy problem for the linear wave equation without source term on the maximally extended Schwarzschild - de Sitter spacetime in the case of non-extremal black-hole corresponding to parameter values

$$0 < M_{bh} < \frac{1}{3\sqrt{\Lambda}},$$

is considered by Dafermos and Rodnianski [8]. They proved that in the region bounded by a set of black/white hole horizons and cosmological horizons, solutions converge pointwise to a constant faster than any given polynomial rate, where the decay is measured with respect to natural future-directed advanced and retarded time coordinates.

There is an important question of local energy decay for the solution of the wave equation and Klein-Gordon equation in black hole spacetime. Results on the decay of local energy can provide a proof of the global nonlinear stability of the spacetime. The global nonlinear stability we believe is only known for Minkowski spacetime. Bony and Hafner [4] describe an expansion of the solution of the wave equation in the Schwarzschild - de Sitter metric in terms of resonances. The resonances correspond to the frequencies and rates of dumping of signals emitted by the black hole in the presence of perturbations (see [9, Chapter 4.35]). The main term in the expansion obtained in [4] is due to a zero resonance. The error term decays polynomially if one permits a logarithmic derivative loss in the angular directions and exponentially if one permits a small derivative loss in the angular directions.

In the present paper we set $M_{bh} = 0$ to exclude black holes. The case of the Klein-Gordon equation in the presence of the black holes will be discussed in the forthcoming paper. Thus, the de Sitter line element has the form

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right) c^2\, dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)\,.$$

The transformation

$$r' = \frac{r}{\sqrt{1 - r^2/R^2}} e^{-ct/R}\,, \quad t' = t + \frac{R}{2c} \ln\left(1 - \frac{r^2}{R^2}\right)\,, \quad \theta' = \theta\,, \quad \phi' = \phi$$

leads to the following form for the line element:

$$ds^2 = -c^2\, dt'^2 + e^{2ct'/R}(dr'^2 + r'^2\, d\theta'^2 + r'^2 \sin^2\theta'\, d\phi'^2)\,. \tag{0.5}$$

Finally, defining new space coordinates x', y', z' connected with r', θ', ϕ' by the usual equations connecting Cartesian coordinates and polar coordinates in a Euclidean space, (0.5) may be written [22, Sec.134]

$$ds^2 = -c^2\, dt'^2 + e^{2ct'/R}(dx'^2 + dy'^2 + dz'^2)\,. \tag{0.6}$$

The new coordinates x', y', z', t' can take all values from $-\infty$ to ∞. Here R is the "radius" of the universe. The de Sitter model allows us to get an explanation of the actual red shift of spectral lines observed by Hubble and Humanson [22]. The de Sitter model also enjoys the advantage of being the only time-dependent

cosmological model for which both particle creation and vacuum stress have been explicitly evaluated by all known techniques [3]. In a certain sense all solutions look like the de Sitter solution at late times [16].

The homogeneous and isotropic cosmological models possess highest symmetry that makes them more amenable to rigorous study. Among them we mention FLRW (Friedmann-Lematre-Robertson-Walker) models. The simplest class of cosmological models can be obtained if we assume additionally that the metric of the slices of constant time is flat and that the spacetime metric can be written in the form

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

with an appropriate scale factor $a(t)$. Although on the made assumptions, the spatially flat FLRW models appear to give a good explanation of our universe. The assumption that the universe is expanding leads to the positivity of the time derivative $\frac{d}{dt}a(t)$. A further assumption that the universe obeys the accelerated expansion suggests that the second derivative $\frac{d^2}{dt^2}a(t)$ is positive. A substantial amount of the observational material can be satisfactorily interpreted in terms of the models which take into account existing acceleration of the recession of distant galaxies.

The time dependence of the function $a(t)$ is determined by the Einstein field equations for gravity. The Einstein equations with the *cosmological constant* Λ have form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu},$$

where term $\Lambda g_{\mu\nu}$ can be interpreted as an energy-momentum of the vacuum. Even a small value of Λ could have drastic effects on the evolution of the universe. Under the assumption of FLRW symmetry the equation of motion in the case of positive cosmological constant Λ leads to the solution

$$a(t) = a(0)e^{\sqrt{\frac{\Lambda}{3}}t},$$

which produces models with exponentially accelerated expansion. The model described by the last equation is usually called the *de Sitter model*.

The unknown of principal importance in the Einstein equations is a metric g. It comprises the basic geometrical feature of the gravitational field, and consequently explains the phenomenon of the mutual gravitational attraction of substance. In the presence of matter these equations contain a non-vanishing right hand side $-8\pi G T_{\mu\nu}$. In general the matter fields described by the function ϕ must satisfy equations of motion and in the case of the massive scalar field the equation of motion is that ϕ should satisfy the Klein-Gordon equation generated by the metric g. In the de Sitter universe the equation for the scalar field with mass m and potential function V written out explicitly in coordinates is (See, e.g. [14, 26].)

$$\phi_{tt} + nH\phi_t - e^{-2Ht} \bigtriangleup \phi + m^2\phi = -V'(\phi)\,. \tag{0.7}$$

Here $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $\bigtriangleup$ is the Laplace operator on the flat metric, $\bigtriangleup := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$, while $H = \sqrt{\Lambda/3}$ is Hubble constant. If we introduce new unknown function $u = e^{-\frac{n}{2}Ht}\phi$, then the semilinear Klein-Gordon equation for u on de Sitter spacetime takes the form

$$u_{tt} - e^{-2Ht} \bigtriangleup u + M^2 u = -e^{\frac{n}{2}t}V'(e^{-\frac{n}{2}t}u), \tag{0.8}$$

where $M^2 := m^2 - \frac{n^2}{4}H^2$. Henceforth the quantity M, with nonnegative real part $\Re M \geq 0$, defined by

$$M^2 := m^2 - \frac{n^2}{4}H^2\,,$$

will be called the "curved mass" of particle. We extract a linear part of the equation (0.8) as an initial model that must be treated first:

$$u_{tt} - e^{-2Ht} \bigtriangleup u + M^2 u = 0\,. \tag{0.9}$$

The fundamental solutions and the Cauchy problem for the equation with $M = 0$ in the backward direction of time are considered in [33].

The de Sitter line element in the higher dimensional analogue of de Sitter space is

$$ds^2 = -dt^2 + e^{2Ht}\big((dx^1)^2 + \ldots + (dx^n)^2\big)\,.$$

It is a simplified version of the multidimensional cosmological models with the metric tensor given by

$$g = -e^{2\gamma(t)}dt^2 + e^{2\phi^1(t)}g_1 + \ldots + e^{2\phi^n(t)}g_n\,,$$

and can be chosen as a starting point for the study. The multidimensional cosmological models have attracted a lot of attention during recent years in constructing mathematical models of an anisotropic universe (see, e.g. [7, 16] and references therein).

The equation (0.9) is strictly hyperbolic. That implies the well-posedness of the Cauchy problem for (0.9) in the different functional spaces. The coefficient of the equation is an analytic function and Holmgren's theorem implies a local uniqueness in the space of distributions. Moreover, the speed of propagation is finite, namely, it is equal to e^{-Ht} for every $t \in \mathbb{R}$. The second-order strictly hyperbolic equation (0.9) possesses two fundamental solutions resolving the Cauchy problem. They can be written microlocally in terms of the Fourier integral operators [17], which give a complete description of the wave front sets of the solutions. The distance between two characteristic roots $\lambda_1(t,\xi)$ and $\lambda_2(t,\xi)$ of the equation (0.9) is $|\lambda_1(t,\xi) - \lambda_2(t,\xi)| = e^{-Ht}|\xi|$, $t \in \mathbb{R}$, $\xi \in \mathbb{R}^n$. It tends to zero as t approaches ∞. Thus, the operator is not uniformly (that is for all $t \in \mathbb{R}$) strictly hyperbolic. Moreover, the finite integrability of the characteristic roots, $\int_0^\infty |\lambda_i(t,\xi)|dt < \infty$, leads to the existence of so-called "horizon" for that equation. More precisely, any signal emitted from the spatial point $x_0 \in \mathbb{R}^n$ at time $t_0 \in \mathbb{R}$ remains inside the ball $|x-x_0| < \frac{1}{H}e^{-Ht_0}$ for all time $t \in (t_0,\infty)$. The equation (0.9) is neither Lorentz invariant nor invariant with respect to usual scaling and that brings additional difficulties. In particular, it can cause a nonexistence of the $L^p - L^q$ decay for the solutions in the direction of time. In [30] it is mentioned the model equation with permanently bounded domain of influence, power decay of characteristic roots, and without $L^p - L^q$ decay for the solutions that illustrates that phenomenon. The above mentioned $L^p - L^q$ decay estimates are one of the important tools for studying nonlinear equations (see, e.g. [25, 27]).

The time inversion transformation $t \to -t$ reduces the equation (0.9) to the mathematically equivalent equation

$$u_{tt} - e^{2Ht} \bigtriangleup u + M^2 u = 0\,. \tag{0.10}$$

The wave equation, that is equation (0.10) with $M = 0$, was investigated in [15] by the second author. More precisely, in [15] the resolving operator for the Cauchy problem

$$u_{tt} - e^{2t} \bigtriangleup u = 0, \qquad u(x,0) = \varphi_0(x), \quad u_t(x,0) = \varphi_1(x)\,, \tag{0.11}$$

is written as a sum of the Fourier integral operators with the amplitudes given in terms of the Bessel functions and in terms of confluent hypergeometric functions. In particular, it is proved in [15] that for $t > 0$ the solution of the Cauchy problem (0.11) is given by

$$\begin{aligned}
u(x,t) \quad = \quad & -i\frac{2}{(2\pi)^n}\int_{\mathbb{R}^n}\Big\{e^{i[x\cdot\xi+(e^t-1)|\xi|]}H_+\big(\frac{1}{2};1;2ie^t|\xi|\big)H_-\big(\frac{3}{2};3;2i|\xi|\big)\\
&\qquad - e^{i[x\cdot\xi-(e^t-1)|\xi|]}H_-\big(\frac{1}{2};1;2ie^t|\xi|\big)H_+\big(\frac{3}{2};3;2i|\xi|\big)\Big\}|\xi|^2\hat{\varphi}_0(\xi)d\xi\\
& -i\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\Big\{e^{i[x\cdot\xi+(e^t-1)|\xi|]}H_+\big(\frac{1}{2};1;2ie^t|\xi|\big)H_-\big(\frac{1}{2};1;2i|\xi|\big)\\
&\qquad - e^{i[x\cdot\xi-(e^t-1)|\xi|]}H_-\big(\frac{1}{2};1;2ie^t|\xi|\big)H_+\big(\frac{1}{2};1;2i|\xi|\big)\Big\}\hat{\varphi}_1(\xi)d\xi\,.
\end{aligned}$$

In the notations of [2] the last functions are $H_-(\alpha;\gamma;z) = e^{i\alpha\pi}\Psi(\alpha;\gamma;z)$ and $H_+(\alpha;\gamma;z) = e^{i\alpha\pi}\Psi(\gamma-\alpha;\gamma;-z)$, where function $\Psi(a;c;z)$ is defined in [2, Sec.6.5]. Here $\hat{\varphi}(\xi)$ is a Fourier transform of $\varphi(x)$.

The typical $L^p - L^q$ decay estimates obtained in [15] by dyadic decomposition of the phase space contain some derivative loss. More precisely, it is proved that for the solution $u = u(x,t)$ to the Cauchy problem (0.11) with $n \geq 2$, $\varphi_0(x) \in C_0^\infty(\mathbb{R}^n)$ and $\varphi_1(x) = 0$ for all large $t \geq T > 0$, the following estimate is satisfied

$$\|u(x,t)\|_{L^q(\mathbb{R}^n)} \leq C(1+e^t)^{-\frac{1}{2}(n-1)(\frac{1}{p}-\frac{1}{q})}\|\varphi_0\|_{W_p^N(\mathbb{R}^n)}, \tag{0.12}$$

where $1 < p \le 2$, $\frac{1}{p}+\frac{1}{q}=1$, and $\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q}) \le N < \frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q})+1$ and $W_p^N(\mathbb{R}^n)$ is the Sobolev space. In particular, the derivative loss, N, is positive, unless $p=q=2$. This derivative loss phenomenon exists for the classical wave equation as well. Indeed, it is well-known (see, e.g., [19, 20, 24]) that for the Cauchy problem $u_{tt}-\triangle u=0,\ u(x,0)=\varphi(x),\ u_t(x,0)=0$, the estimate $\|u(x,t)\|_{L^q(\mathbb{R}^n_x)} \le C\|\varphi(x)\|_{L^q(\mathbb{R}^n_x)}$ fails to fulfill even for small positive t unless $q=2$. The obstacle is created by the distinguishing feature of the (different from translation) Fourier integral operators of order zero, which compose a resolving operator.

According to Theorem 1 [15], for the solution $u=u(x,t)$ to the Cauchy problem (0.11) with $n \ge 2$, $\varphi_0(x)=0$ and $\varphi_1(x) \in C_0^\infty(\mathbb{R}^n)$ for all large $t \ge T > 0$ and for any small $\varepsilon > 0$, the following estimate is satisfied

$$\|u(x,t)\|_{L^q(\mathbb{R}^n)} \le C_\varepsilon(1+t)(1+e^t)^{r_0-n(\frac{1}{p}-\frac{1}{q})}\|\varphi_1\|_{W_p^N(\mathbb{R}^n)},$$

where $1 < p \le 2$, $\frac{1}{p}+\frac{1}{q}=1$, $r_0=\max\{\varepsilon;\frac{(n+1)}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{q}\}$, $\frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{q} \le N < \frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})+\frac{1}{p}$.

The nonlinear equations (0.7) and (0.8) are those we would like to solve, but the linear problem is a natural first step. Exceptionally efficient tool for the studying nonlinear equations is a fundamental solution of the associate linear operator.

The fundamental solutions for the operator of the equation (0.11) are constructed in [33] and the representations of the solutions of the Cauchy problem

$$u_{tt}-e^{2t}\triangle u=f(x,t), \qquad u(x,0)=\varphi_0(x), \quad u_t(x,0)=\varphi_1(x)\,, \tag{0.13}$$

are given in the terms of the solutions of wave equation in Minkowski spacetime. Then in [33] for $n \ge 2$ the following decay estimate

$$\begin{aligned}\|(-\triangle)^{-s}u(x,t)\|_{L^q(\mathbb{R}^n)} \le{}& Ce^{t(2s-n(\frac{1}{p}-\frac{1}{q}))}\int_0^t(1+t-b)\|f(x,b)\|_{L^p(\mathbb{R}^n)}\,db\\ &+C(e^t-1)^{2s-n(\frac{1}{p}-\frac{1}{q})}\left\{\|\varphi_0(x)\|_{L^p(\mathbb{R}^n)}+\|\varphi_1(x)\|_{L^p(\mathbb{R}^n)}(1+t)(1-e^{-t})\right\}\end{aligned}$$

is proven, provided that $s \ge 0$, $1 < p \le 2$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{2}(n+1)\left(\frac{1}{p}-\frac{1}{q}\right) \le 2s \le n\left(\frac{1}{p}-\frac{1}{q}\right) < 2s+1$. Moreover, this estimate is fulfilled for $n=1$ and $s=0$ as well as if $\varphi_0(x)=0$ and $\varphi_1(x)=0$. Case of $n=1$, $f(x,t)=0$, and non-vanishing $\varphi_1(x)$ and $\varphi_1(x)$ also is discussed in Section 8 [33].

In the construction of the fundamental solutions for the operator (0.9) we follow the approach proposed in [29] that allows us to represent the fundamental solutions as some integral of the family of the fundamental solutions of the Cauchy problem for the wave equation without source term. The kernel of that integral contains the Gauss's hypergeometric function. In that way, many properties of the wave equation can be extended to the hyperbolic equations with the time dependent speed of propagation. That approach was successfully applied in [31, 32] by the first author to investigate the semilinear Tricomi-type equations.

Thus, in the present paper we consider Klein-Gordon operator in de Sitter model of the universe, that is

$$\mathcal{S} := \partial_t^2 - e^{-2t}\triangle + M^2\,,$$

where $M \ge 0$ is the reduced mass, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. We look for the fundamental solution $E=E(x,t;x_0,t_0)$,

$$E_{tt}-e^{-2t}\Delta E - M^2E=\delta(x-x_0,t-t_0),$$

with a support in the "forward light cone" $D_+(x_0,t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, and for the fundamental solution with a support in the "backward light cone" $D_-(x_0,t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, defined as follows

$$D_\pm(x_0,t_0) \quad := \quad \left\{(x,t) \in \mathbb{R}^{n+1}\,;\, |x-x_0| \le \pm(e^{-t_0}-e^{-t})\right\}. \tag{0.14}$$

In fact, any intersection of $D_-(x_0,t_0)$ with the hyperplane $t = \mathit{const} < t_0$ determines the so-called dependence domain for the point (x_0,t_0), while the intersection of $D_+(x_0,t_0)$ with the hyperplane $t=\mathit{const} > t_0$ is the so-called domain of influence of the point (x_0,t_0). The equation (0.9) is non-invariant with respect to time

inversion. Moreover, the dependence domain is wider than any given ball if time *const* $> t_0$ is sufficiently large, while the domain of influence is permanently, for all time *const* $< t_0$, in the ball of the radius e^{t_0}.

Define for $t_0 \in \mathbb{R}$ in the domain $D_+(x_0,t_0) \cup D_-(x_0,t_0)$ the function

$$
\begin{aligned}
& E(x,t;x_0,t_0) && (0.15)\\
= \quad & (4e^{-t_0-t})^{iM}\Big((e^{-t}+e^{-t_0})^2-(x-x_0)^2\Big)^{-\frac{1}{2}-iM} F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-t_0}-e^{-t})^2-(x-x_0)^2}{(e^{-t_0}+e^{-t})^2-(x-x_0)^2}\Big),
\end{aligned}
$$

where $F\big(a,b;c;\zeta\big)$ is the hypergeometric function (See, e.g. [2].). Let $E(x,t;0,b)$ be function (0.15), and set

$$
\mathcal{E}_+(x,t;0,t_0) := \begin{cases} E(x,t;0,t_0) & \text{in } D_+(0,t_0), \\ 0 & \text{elsewhere} \end{cases}, \qquad \mathcal{E}_-(x,t;0,t_0) := \begin{cases} E(x,t;0,t_0) & \text{in } D_-(0,t_0), \\ 0 & \text{elsewhere} \end{cases}.
$$

Since function $E = E(x,t;0,t_0)$ is smooth in $D_\pm(0,t_0)$ and is locally integrable, it follows that $\mathcal{E}_+(x,t;0,\ t_0)$ and $\mathcal{E}_-(x,t;0,\ t_0)$ are distributions whose supports are in $D_+(0,t_0)$ and $D_-(0,t_0)$, respectively. The next theorem gives our first result.

Theorem 0.1 *Suppose that $n=1$. The distributions $\mathcal{E}_+(x,t;0,t_0)$ and $\mathcal{E}_-(x,t;0,t_0)$ are the fundamental solutions for the operator $\mathcal{S} = \partial_t^2 - e^{-2t}\bigtriangleup + M^2$ relative to the point $(0,t_0)$, that is*

$$
\mathcal{S}\mathcal{E}_\pm(x,t;0,t_0) = \delta(x,t-t_0),
$$

or

$$
\frac{\partial^2}{\partial t^2}\mathcal{E}_\pm(x,t;0,t_0) - e^{-2t}\frac{\partial^2}{\partial x^2}\mathcal{E}_\pm(x,t;0,t_0) + M^2\mathcal{E}_\pm(x,t;0,t_0) = \delta(x,t-t_0).
$$

To motivate one construction for the higher dimensional case $n \geq 2$ we follow the approach suggested in [29] and represent fundamental solution $\mathcal{E}_+(x,t;0,t_0)$ as follows

$$
\begin{aligned}
\mathcal{E}_+(x,t;0,t_0) \quad = \quad & \int_{e^{-t}-e^{-t_0}}^{e^{-t_0}-e^{-t}} \mathcal{E}^{string}(x,r)(4e^{-t_0-t})^{iM}\Big((e^{-t_0}+e^{-t})^2-r^2\Big)^{-\frac{1}{2}-iM}\\
& \times F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-t_0}-e^{-t})^2-r^2}{(e^{-t_0}+e^{-t})^2-r^2}\Big)\,dr, \quad t>t_0,
\end{aligned}
$$

where the distribution $\mathcal{E}^{string}(x,t)$ is the fundamental solution of the Cauchy problem for the string equation:

$$
\frac{\partial^2}{\partial t^2}\mathcal{E}^{string} - \frac{\partial^2}{\partial x^2}\mathcal{E}^{string} = 0, \qquad \mathcal{E}^{string}(x,0) = \delta(x),\ \ \mathcal{E}_t^{string}(x,0) = 0\,.
$$

Hence, $\mathcal{E}^{string}(x,t) = \frac{1}{2}\{\delta(x+t)+\delta(x-t)\}$. The integral makes sense in the topology of the space of distributions. The fundamental solution $\mathcal{E}_-(x,t;0,t_0)$ for $t<t_0$ admits a similar representation.

We appeal to the wave equation in Minkowski spacetime to obtain in the next theorem very similar representations of the fundamental solutions of the higher dimensional equation in de Sitter spacetime with $n \geq 2$.

Theorem 0.2 *If $x \in \mathbb{R}^n$, $n \geq 2$, then for the operator $\mathcal{S} = \partial_t^2 - e^{-2t}\bigtriangleup + M^2$ the fundamental solution $\mathcal{E}_{+,n}(x,t;x_0,t_0)$ $(= \mathcal{E}_{+,n}(x-x_0,t;0,t_0))$ with a support in the forward cone $D_+(x_0,t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, $supp\,\mathcal{E}_{+,n} \subseteq D_+(x_0,t_0)$, is given by the following integral $(t>t_0)$*

$$
\begin{aligned}
\mathcal{E}_{+,n}(x-x_0,t;0,t_0) \quad = \quad & 2\int_0^{e^{-t_0}-e^{-t}} \mathcal{E}^w(x-x_0,r)(4e^{-t_0-t})^{iM}\Big((e^{-t_0}+e^{-t})^2-r^2\Big)^{-\frac{1}{2}-iM}\\
& \times F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-t_0}-e^{-t})^2-r^2}{(e^{-t_0}+e^{-t})^2-r^2}\Big)\,dr. \qquad (0.16)
\end{aligned}
$$

Here the function $\mathcal{E}^w(x,t;b)$ is a fundamental solution to the Cauchy problem for the wave equation

$$
\mathcal{E}_{tt}^w - \bigtriangleup\mathcal{E}^w = 0\,, \quad \mathcal{E}^w(x,0) = \delta(x)\,, \quad \mathcal{E}_t^w(x,0) = 0\,.
$$

The fundamental solution $\mathcal{E}_{-,n}(x,t;x_0,t_0)$ $(= \mathcal{E}_{-,n}(x-x_0,t;0,t_0))$ with a support in the backward cone $D_-(x_0,t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, $supp\, \mathcal{E}_{-,n} \subseteq D_-(x_0,t_0)$, is given by the following integral ($t < t_0$)

$$\begin{aligned}\mathcal{E}_{+,n}(x-x_0,t;0,t_0) \quad &= \quad -2\int_{e^{-t_0}-e^{-t}}^{0} \mathcal{E}^w(x-x_0,r)(4e^{-t_0-t})^{iM}\Big((e^{-t_0}+e^{-t})^2-r^2\Big)^{-\frac{1}{2}-iM} \\ &\qquad \times F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-t_0}-e^{-t})^2-r^2}{(e^{-t_0}+e^{-t})^2-r^2}\Big)\,dr. \qquad (0.17)\end{aligned}$$

In particular, the formula (0.16) shows that Huygens's Principle is not valid for the waves propagating in de Sitter model of the universe. Fields satisfying a wave equation in de Sitter model of the universe can be accompanied by *tails* propagating inside the light cone. This phenomenon will be discussed in the spirit of [28] in the forthcoming paper.

Next we use Theorem 0.1 to solve the Cauchy problem for the one-dimensional equation

$$u_{tt} - e^{-2t}u_{xx} + M^2 u = f(x,t)\,, \qquad t>0\,, \quad x\in\mathbb{R}\,, \qquad (0.18)$$

with vanishing initial data,

$$u(x,0) = u_t(x,0) = 0\,. \qquad (0.19)$$

Theorem 0.3 *Assume that the function f is continuous along with its all second order derivatives, and that for every fixed t it has a compact support, $suppf(\cdot,t) \subset \mathbb{R}$. Then the function $u=u(x,t)$ defined by*

$$\begin{aligned}u(x,t) \quad &= \quad \int_0^t db\int_{x-(e^{-b}-e^{-t})}^{x+e^{-b}-e^{-t}} dy\, f(y,b)(4e^{-b-t})^{iM}\Big((e^{-t}+e^{-b})^2-(x-y)^2\Big)^{-\frac{1}{2}-iM} \\ &\qquad \times F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-b}-e^{-t})^2-(x-y)^2}{(e^{-b}+e^{-t})^2-(x-y)^2}\Big)\end{aligned}$$

is a C^2-solution to the Cauchy problem for the equation (0.18) with vanishing initial data, (0.19).

The representation of the solution of the Cauchy problem for the one-dimensional case ($n=1$) of the equation (0.9) without source term is given by the next theorem.

Theorem 0.4 *The solution $u=u(x,t)$ of the Cauchy problem*

$$u_{tt} - e^{-2t}u_{xx} + M^2 u = 0\,, \qquad u(x,0) = \varphi_0(x)\,, \qquad u_t(x,0) = \varphi_1(x)\,, \qquad (0.20)$$

with $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R})$ can be represented as follows

$$\begin{aligned}u(x,t) \quad &= \quad \frac{1}{2}e^{\frac{t}{2}}\Big[\varphi_0(x+1-e^{-t})+\varphi_0(x-1+e^{-t})\Big] + \int_0^{1-e^{-t}} \Big[\varphi_0(x-z)+\varphi_0(x+z)\Big]K_0(z,t)\,dz \\ &\quad + \int_0^{1-e^{-t}} \Big[\varphi_1(x-z)+\varphi_1(x+z)\Big]K_1(z,t)dz\,,\end{aligned}$$

where the kernels $K_0(z,t)$ and $K_1(z,t)$ are defined by

$$\begin{aligned}K_0(z,t) \quad &:= \quad -\Big[\frac{\partial}{\partial b}E(z,t;0,b)\Big]_{b=0} \\ &= \quad (4e^{-t})^{iM}\big((e^{-t}+1)^2-z^2\big)^{-iM}\frac{1}{[(1-e^{-t})^2-z^2]\sqrt{(e^{-t}+1)^2-z^2}} \\ &\quad \times\Bigg[\big(e^{-t}-1-iM(e^{-2t}-1-z^2)\big)F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(1-e^{-t})^2-z^2}{(1+e^{-t})^2-z^2}\Big) \\ &\qquad +\big(1-e^{-2t}+z^2\big)\Big(\frac{1}{2}-iM\Big)F\Big(-\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(1-e^{-t})^2-z^2}{(1+e^{-t})^2-z^2}\Big)\Bigg]\end{aligned}$$

and

$$\begin{aligned} K_1(z,t) &:= E(z,t;0,0) \\ &= (4e^{-t})^{iM}\big((1+e^{-t})^2-z^2\big)^{-\frac{1}{2}-iM} F\left(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(1-e^{-t})^2-z^2}{(1+e^{-t})^2-z^2}\right),\ 0\le z\le 1-e^{-t}, \end{aligned}$$

respectively.

The kernels $K_0(z,t)$ and $K_1(z,t)$ play leading roles in the derivation of $L^p - L^q$ estimates. Their main properties are listed and proved in Section 8.

Next we turn to the higher-dimensional equation with $n \ge 2$.

Theorem 0.5 *If n is odd, $n = 2m+1$, $m \in \mathbb{N}$, then the solution $u = u(x,t)$ to the Cauchy problem*

$$u_{tt} - e^{-2t}\Delta u + M^2 u = f, \quad u(x,0) = 0, \quad u_t(x,0) = 0, \tag{0.21}$$

with $f \in C^\infty(\mathbb{R}^{n+1})$ and with the vanishing initial data is given by the next expression

$$\begin{aligned} u(x,t) &= 2\int_0^t db \int_0^{e^{-b}-e^{-t}} dr_1 \left(\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{\frac{n-3}{2}} \frac{r^{n-2}}{\omega_{n-1}c_0^{(n)}}\int_{S^{n-1}} f(x+ry,b)\,dS_y\right)_{r=r_1} \\ &\quad \times (4e^{-b-t})^{iM}\left((e^{-t}+e^{-b})^2-r_1^2\right)^{-\frac{1}{2}-iM} F\left(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-b}-e^{-t})^2-r_1^2}{(e^{-b}+e^{-t})^2-r_1^2}\right), \end{aligned} \tag{0.22}$$

where $c_0^{(n)} = 1\cdot 3\cdot \ldots \cdot (n-2)$. Constant ω_{n-1} is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

If n is even, $n = 2m$, $m \in \mathbb{N}$, then the solution $u = u(x,t)$ is given by the next expression

$$\begin{aligned} u(x,t) &= 2\int_0^t db \int_0^{e^{-b}-e^{-t}} dr_1 \left(\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{\frac{n-2}{2}} \frac{2r^{n-1}}{\omega_{n-1}c_0^{(n)}}\int_{B_1^n(0)} \frac{f(x+ry,b)}{\sqrt{1-|y|^2}}\,dV_y\right)_{r=r_1} \\ &\quad \times (4e^{-b-t})^{iM}\left((e^{-t}+e^{-b})^2-r_1^2\right)^{-\frac{1}{2}-iM} F\left(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-b}-e^{-t})^2-r_1^2}{(e^{-b}+e^{-t})^2-r_1^2}\right). \end{aligned} \tag{0.23}$$

Here $B_1^n(0) := \{|y| \le 1\}$ is the unit ball in $\mathbb{R}^n$, while $c_0^{(n)} = 1\cdot 3\cdot \ldots \cdot (n-1)$.

Thus, in both cases, of even and odd n, one can write

$$\begin{aligned} u(x,t) &= 2\int_0^t db \int_0^{e^{-b}-e^{-t}} dr\, v(x,r;b)(4e^{-b-t})^{iM}\left((e^{-t}+e^{-b})^2-r^2\right)^{-\frac{1}{2}-iM} \\ &\quad \times F\left(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2}\right), \end{aligned} \tag{0.24}$$

where the function $v(x,t;b)$ is a solution to the Cauchy problem for the wave equation

$$v_{tt} - \triangle v = 0\,, \quad v(x,0;b) = f(x,b)\,, \quad v_t(x,0;b) = 0\,.$$

The next theorem gives representation of the solutions of equation (0.9) with the initial data prescribed at $t = 0$.

Theorem 0.6 *The solution $u = u(x,t)$ to the Cauchy problem*

$$u_{tt} - e^{-2t} \triangle u + M^2 u = 0\,, \quad u(x,0) = \varphi_0(x)\,, \quad u_t(x,0) = \varphi_1(x)\,, \tag{0.25}$$

with φ_0, $\varphi_1 \in C_0^\infty(\mathbb{R}^n)$, $n \ge 2$, can be represented as follows:

$$\begin{aligned} u(x,t) &= e^{\frac{t}{2}} v_{\varphi_0}(x,\phi(t)) + 2\int_0^1 v_{\varphi_0}(x,\phi(t)s)K_0(\phi(t)s,t)\phi(t)\,ds \\ &\quad + 2\int_0^1 v_{\varphi_1}(x,\phi(t)s)K_1(\phi(t)s,t)\phi(t)\,ds, \quad x \in \mathbb{R}^n,\ t > 0\,, \end{aligned} \tag{0.26}$$

$\phi(t) := 1-e^{-t}$, by means of the kernels K_0 and K_1 have been defined in Theorem 0.4. Here for $\varphi \in C_0^\infty(\mathbb{R}^n)$ and for $x \in \mathbb{R}^n$, $n = 2m+1$, $m \in \mathbb{N}$,

$$v_\varphi(x,\phi(t)s) := \left(\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{\frac{n-3}{2}} \frac{r^{n-2}}{\omega_{n-1}c_0^{(n)}} \int_{S^{n-1}} \varphi(x+ry)\, dS_y \right)_{r=\phi(t)s}$$

while for $x \in \mathbb{R}^n$, $n = 2m$, $m \in \mathbb{N}$,

$$v_\varphi(x,\phi(t)s) := \left(\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{\frac{n-2}{2}} \frac{2r^{n-1}}{\omega_{n-1}c_0^{(n)}} \int_{B_1^n(0)} \frac{1}{\sqrt{1-|y|^2}}\varphi(x+ry)\, dV_y \right)_{r=s\phi(t)} .$$

The function $v_\varphi(x,\phi(t)s)$ coincides with the value $v(x,\phi(t)s)$ of the solution $v(x,t)$ of the Cauchy problem

$$v_{tt} - \triangle v = 0, \quad v(x,0) = \varphi(x), \quad v_t(x,0) = 0\,.$$

As a consequence of the above theorems we obtain in Sections 9-10 for $n \geq 2$ and for the particles with "large" mass m, $m \geq n/2$, that is, with nonnegative curved mass $M \geq 0$, the following $L^p - L^q$ estimate

$$\begin{aligned} \|(-\triangle)^{-s}u(x,t)\|_{L^q(\mathbb{R}^n)} \leq C\int_0^t \|f(x,b)\|_{L^p(\mathbb{R}^n)} e^{-b}\left(e^{-b}-e^{-t}\right)^{1+2s-n(\frac{1}{p}-\frac{1}{q})}(1+t-b)\, db \\ +C(1+t)(1-e^{-t})^{2s-n(\frac{1}{p}-\frac{1}{q})}\Big\{e^{\frac{t}{2}}\|\varphi_0(x)\|_{L^p(\mathbb{R}^n)} + (1-e^{-t})\|\varphi_1\|_{L^p(\mathbb{R}^n)}\Big\} \end{aligned} \quad (0.27)$$

provided that $s \geq 0$, $1 < p \leq 2$, $\frac{1}{p}+\frac{1}{q} = 1$, $\frac{1}{2}(n+1)\left(\frac{1}{p}-\frac{1}{q}\right) \leq 2s \leq n\left(\frac{1}{p}-\frac{1}{q}\right) < 2s+1$. Moreover, according to Theorem 7.1 the estimate (0.27) is valid for $n = 1$ and $s = 0$ as well as if $\varphi_0(x) = 0$ and $\varphi_1(x) = 0$. Case of $n = 1$, $f(x,t) = 0$, and non-vanishing $\varphi_1(x)$ and $\varphi_1(x)$ is discussed in Section 8. The case of particles with small mass $m < n/2$ will be discussed in the forthcoming paper.

The paper is organized as follows. In Section 1 we construct the Riemann function of the operator of (0.9) in the characteristic coordinates for the case of $n = 1$. That Riemann function used in Section 2 to prove Theorem 0.1. Then in Section 3 we apply the fundamental solutions to solve the Cauchy problem with the source term and with the vanishing initial data given at $t = 0$. More precisely, we give a representation formula for the solutions. In that section we also prove several basic properties of the function $E(x,t;y,b)$. In Sections 4-5 we use formulas of Section 3 to derive and to complete the list of representation formulas for the solutions of the Cauchy problem for the case of one-dimensional spatial variable. The higher-dimensional equation with the source term is considered in Section 6, where we derive a representation formula for the solutions of the Cauchy problem with the source term and with the vanishing initial data given at $t = 0$. In the same section this formula is used to derive the fundamental solutions of the operator and to complete the proof of Theorem 0.6. Then in Sections 7-10 we establish the $L^p - L^q$ decay estimates. Applications of all these results to the nonlinear equations will be done in the forthcoming paper.

1 The Riemann Function

In the characteristic coordinates l and m,

$$l = x + e^{-t}, \qquad m = x - e^{-t}, \quad (1.1)$$

one has

$$\frac{\partial^2}{\partial t^2} = \frac{1}{2}(l-m)\left(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\right) + \frac{1}{4}(l-m)^2\left(\frac{\partial^2}{\partial l^2} - 2\frac{\partial^2}{\partial l\,\partial m} + \frac{\partial^2}{\partial m^2}\right)$$

and

$$e^{-2t}\frac{\partial^2}{\partial x^2} = \frac{1}{4}(l-m)^2\left(\frac{\partial^2}{\partial l^2} + 2\frac{\partial^2}{\partial l\,\partial m} + \frac{\partial^2}{\partial m^2}\right).$$

Then the operator $\mathcal{S}$ of the equation (0.18) reads

$$\mathcal{S} := \frac{\partial^2}{\partial t^2} - e^{-2t}\frac{\partial^2}{\partial x^2} + M^2 = -(l-m)^2\left\{\frac{\partial^2}{\partial l\,\partial m} - \frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big) - \frac{1}{(l-m)^2}M^2\right\}. \tag{1.2}$$

In particular, in the new variables the equation

$$\Big(\frac{\partial^2}{\partial t^2} - e^{-2t}\frac{\partial^2}{\partial x^2} + M^2\Big)u = 0 \quad \text{implies} \quad \left\{\frac{\partial^2}{\partial l\,\partial m} - \frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big)\right\}u - \frac{1}{(l-m)^2}M^2 u = 0\,.$$

We need the following lemma with $\gamma = \frac{1}{2} + iM$.

Lemma 1.1 *The function*

$$V(l,m;a,b) = (l-b)^{-\gamma}(a-m)^{-\gamma}F\Big(\gamma,\gamma;1;\frac{(l-a)(m-b)}{(l-b)(m-a)}\Big)$$

solves the equation

$$\left\{\frac{\partial^2}{\partial l\,\partial m} - \frac{1}{(l-m)}\gamma\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big)\right\}V(l,m;a,b) = 0\,. \tag{1.3}$$

Proof. We denote the argument of the hypergeometric function by z, and evaluate its derivatives,

$$z := \frac{(l-a)(m-b)}{(l-b)(m-a)}\,, \qquad \frac{\partial}{\partial l}z = \frac{(a-b)(b-m)}{(l-b)^2(a-m)}\,, \quad \frac{\partial}{\partial m}z = -\frac{(a-b)(a-l)}{(b-l)(a-m)^2}\,.$$

Further, we obtain

$$\frac{\partial}{\partial l}V(l,m;a,b) = (a-m)^{-\gamma}(l-b)^{-\gamma-1}\left\{-\gamma F\Big(\gamma,\gamma;1;z\Big) + \frac{(a-b)(b-m)}{(l-b)(a-m)}F'_z\Big(\gamma,\gamma;1;z\Big)\right\}.$$

Next

$$\frac{\partial}{\partial m}V(l,m;a,b) = (a-m)^{-\gamma}(l-b)^{-\gamma}\left\{\gamma(a-m)^{-1}F\Big(\gamma,\gamma;1;z\Big) - \frac{(a-b)(a-l)}{(b-l)(a-m)^2}F'_z\Big(\gamma,\gamma;1;z\Big)\right\}.$$

Then

$$\begin{aligned}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big)V(l,m;a,b) \quad &= \quad (a-m)^{-\gamma}(l-b)^{-\gamma}\,(-\gamma)\,F\big(\gamma,\gamma;1;z\big)\Big\{\frac{1}{(l-b)} + \frac{1}{(a-m)}\Big\} \\ &\quad +(a-m)^{-\gamma-1}(l-b)^{-\gamma-1}F'_z\big(\gamma,\gamma;1;z\big)(a-b)\left\{\frac{(b-m)}{(l-b)} - \frac{(a-l)}{(a-m)}\right\}.\end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{\partial^2}{\partial l\,\partial m}V(l,m;a,b) \quad &= \quad (a-m)^{-\gamma-1}\left[-\gamma(l-b)^{-\gamma-1}\left\{\gamma F\Big(\gamma,\gamma;1;z\Big) - \frac{(a-b)(a-l)}{(b-l)(a-m)}F'_z\Big(\gamma,\gamma;1;z\Big)\right\}\right] \\ &\quad + (a-m)^{-\gamma-1}\left[(l-b)^{-\gamma}\frac{\partial}{\partial l}\left\{\gamma F\Big(\gamma,\gamma;1;z\Big) - \frac{(a-b)(a-l)}{(b-l)(a-m)}F'_z\Big(\gamma,\gamma;1;z\Big)\right\}\right].\end{aligned}$$

We calculate

$$\begin{aligned}&\frac{\partial}{\partial l}\left\{\gamma F\Big(\gamma,\gamma;1;z\Big) - \frac{(a-b)(a-l)}{(b-l)(a-m)}F'_z\Big(\gamma,\gamma;1;z\Big)\right\} \\ = \quad &\gamma F'_z\Big(\gamma,\gamma;1;z\Big)\frac{(a-b)(b-m)}{(l-b)^2(a-m)} - \frac{(a-b)^2}{(b-l)^2(a-m)}F'_z\Big(\gamma,\gamma;1;z\Big) \\ &-\frac{(a-b)(a-l)}{(b-l)(a-m)}F_{zz}{''}\Big(\gamma,\gamma;1;z\Big)\frac{(a-b)(b-m)}{(l-b)^2(a-m)}\,.\end{aligned}$$

Here $\frac{\partial}{\partial l}\frac{(a-b)(a-l)}{(b-l)(a-m)} = \frac{(a-b)^2}{(b-l)^2(a-m)}$. Finally,

$$\begin{aligned}\frac{\partial^2}{\partial l\,\partial m}V(l,m;a,b) \;&=\; (a-m)^{-\gamma-1}\Big[-\gamma(l-b)^{-\gamma-1}\Big\{\gamma F\Big(\gamma,\gamma;1;z\Big) - \frac{(a-b)(a-l)}{(b-l)(a-m)}F_z'\Big(\gamma,\gamma;1;z\Big)\Big\}\Big] \\ &+ (a-m)^{-\gamma-1}(l-b)^{-\gamma}\Big\{\gamma F_z'\Big(\gamma,\gamma;1;z\Big)\frac{(a-b)(b-m)}{(l-b)^2(a-m)} \\ &- \frac{(a-b)^2}{(b-l)^2(a-m)}F_z'\Big(\gamma,\gamma;1;z\Big) - \frac{(a-b)(a-l)}{(b-l)(a-m)}F_{zz}''\Big(\gamma,\gamma;1;z\Big)\frac{(a-b)(b-m)}{(l-b)^2(a-m)}\Big\}.\end{aligned}$$

The coefficients of the derivatives of the hypergeometric function, F_{zz}'', $F_z{}'$, and of F in the expression for $\dfrac{\partial^2}{\partial l\,\partial m}V(l,m;a,b)$ are

$$\begin{aligned}&(a-m)^{-\gamma-3}(l-b)^{-\gamma-3}(a-b)^2(a-l)(b-m)\,,\\ &(a-m)^{-\gamma-2}(l-b)^{-\gamma-2}(a-b)\Big\{\gamma(b-m-a+l)-(a-b)\Big\}\,,\\ &(a-m)^{-\gamma-1}\,(-\gamma)\,(l-b)^{-\gamma-1}\{-(-\gamma)\} \;=\; -(a-m)^{-\gamma-1}(l-b)^{-\gamma-1}\gamma^2\,,\end{aligned}$$

respectively. The coefficients of $F_z{}'$ and F in the expression for $\dfrac{1}{(l-m)}\gamma\left(\dfrac{\partial}{\partial l}-\dfrac{\partial}{\partial m}\right)V(l,m;a,b)$ are

$$\frac{1}{(l-m)}\gamma(a-m)^{-\gamma-1}(l-b)^{-\gamma-1}(a-b)\left\{\frac{(b-m)}{(l-b)}-\frac{(a-l)}{(a-m)}\right\}$$

and

$$-\frac{1}{(l-m)}\gamma(a-m)^{-\gamma}(l-b)^{-\gamma}\gamma\Big\{\frac{1}{(l-b)}+\frac{1}{(a-m)}\Big\}\,.$$

Now we turn to the equation (1.3). The coefficients of F and $F_z{}'$ in that equation are

$$\frac{1}{(m-l)}(a-b)(a-m)^{-\gamma-1}(l-b)^{-\gamma-1}(-\gamma^2),$$

and

$$\begin{aligned}&\frac{1}{(m-l)}(a-m)^{-\gamma-1}(l-b)^{-\gamma-1}(a-b)\\ &\times\left[\gamma\frac{(m-l)}{(a-m)(l-b)}(b-m-a+l)+\gamma\left\{\frac{(b-m)}{(l-b)}-\frac{(a-l)}{(a-m)}\right\}-\frac{(m-l)}{(a-m)(l-b)}(a-b)\right].\end{aligned}$$

The first two terms in the brackets can be written as follows

$$\gamma\frac{(m-l)}{(a-m)(l-b)}(b-m-a+l)+\gamma\left\{\frac{(b-m)}{(l-b)}-\frac{(a-l)}{(a-m)}\right\} = -2\gamma z\,,$$

while the last term can be transformed to

$$-\frac{(m-l)}{(a-m)(l-b)}(a-b) = 1-z\,.$$

Thus, the coefficient of $F_z{}'$ in the equation (1.3) is

$$\frac{1}{(m-l)}(a-m)^{-\gamma-1}(l-b)^{-\gamma-1}(a-b)\left[1-(1+2\gamma)z\right]\,.$$

Finally, the coefficient of F_{zz}'' in the equation (1.3) is

$$\frac{1}{(m-l)}(a-m)^{-\gamma-1}(l-b)^{-\gamma-1}(a-b)\Big[\frac{(a-b)(a-l)(b-m)(m-l)}{(a-m)^2(l-b)^2}\Big]\,,$$

where

$$\frac{(a-b)(a-l)(b-m)(m-l)}{(a-m)^2(l-b)^2} = z(1-z)\,.$$

Hence, the left-hand side of (1.3) reads

$$\begin{aligned}
&\Big\{\frac{\partial^2}{\partial l\,\partial m} - \frac{1}{(l-m)}\gamma\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big)\Big\}V(l,m;a,b)\\
= \ & \frac{1}{(m-l)}(a-m)^{-\gamma-1}(l-b)^{-\gamma-1}(a-b)\Big[z(1-z)F_{zz} + \big(1-(1+2\gamma)z\big)F_z - \gamma^2 F\Big] = 0\,,
\end{aligned}$$

and vanishes, since F solves the Gauss hypergeometic equation with $c=1$, $a=\gamma$, and $b=\gamma$. Lemma is proven. $\square$

Lemma 1.2 *For $\gamma \in \mathbb{C}$ such that $F\big(\gamma,\gamma;1;z\big)$ is well defined, the function*

$$\begin{aligned}
E(l,m;a,b) \ &:= \ (a-b)^{\gamma-\frac{1}{2}}(l-m)^{\gamma-\frac{1}{2}}V(l,m;a,b)\\
&= \ (a-b)^{\gamma-\frac{1}{2}}(l-m)^{\gamma-\frac{1}{2}}(l-b)^{-\gamma}(a-m)^{-\gamma}F\Big(\gamma,\gamma;1;\frac{(l-a)(m-b)}{(l-b)(m-a)}\Big)
\end{aligned}$$

solves the equation

$$\Big\{\frac{\partial^2}{\partial l\,\partial m} - \frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big)\Big\}E(l,m;a,b) + \frac{1}{(l-m)^2}\left(\frac{1}{2}-\gamma\right)^2 E(l,m;a,b) = 0\,. \tag{1.4}$$

Proof. Indeed, straightforward calculations show

$$\begin{aligned}
&(a-b)^{-\gamma+\frac{1}{2}}\left\{\frac{\partial^2}{\partial l\,\partial m} - \frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big) + \frac{1}{(l-m)^2}\left(\frac{1}{2}-\gamma\right)^2\right\}E\\
= \ & (l-m)^{\gamma-\frac{1}{2}}\left[V_{lm} - \frac{1}{(l-m)}\gamma\,(V_l - V_m)\right] = 0\,.
\end{aligned}$$

Lemma is proven. $\square$

Consider now the operator

$$\mathcal{S}_{ch}^* := \frac{\partial^2}{\partial l\,\partial m} + \frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big) - \frac{1}{(l-m)^2}(M^2+1)\,,$$

which is a formally adjoint to the operator

$$\mathcal{S}_{ch} := \frac{\partial^2}{\partial l\,\partial m} - \frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big) - \frac{1}{(l-m)^2}M^2\,.$$

In the next lemma the Riemann function is presented.

Proposition 1.3 *The function*

$$\begin{aligned}
R(l,m;a,b) \ &= \ (l-m)E(l,m;a,b)\\
&= \ (a-b)^{iM}(l-m)^{1+iM}(l-b)^{-\frac{1}{2}-iM}(a-m)^{-\frac{1}{2}-iM}F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(l-a)(m-b)}{(l-b)(m-a)}\Big)
\end{aligned}$$

defined for all l, m, a, $b \in \mathbb{R}$, such that $l>m$, is a unique solution of the equation $\mathcal{S}_{ch}^*R=0$, which satisfies the following conditions:

(i) $R_l = \dfrac{1}{2(l-m)}R$ *along the line $m=b$;*

(ii) $R_m = -\dfrac{1}{2(l-m)}R$ *along the line $l=a$;*

(iii) $R(a,b;a,b)=1$.

Proof. Indeed, if we denote $\gamma = \frac{1}{2} + iM$, then for the Riemann function we have

$$R(l,m;a,b) = (a-b)^{\gamma-\frac{1}{2}}(l-m)^{\gamma+\frac{1}{2}}V(l,m;a,b) = (l-m)E(l,m;a,b)\,.$$

The operators $\mathcal{S}_{ch}$ and $\mathcal{S}^*_{ch}$ can be written as follows:

$$\begin{aligned}
\mathcal{S}_{ch} &= \frac{\partial^2}{\partial l\,\partial m} - \frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big) + \frac{1}{(l-m)^2}\left(\gamma - \frac{1}{2}\right)^2 ,\\
\mathcal{S}^*_{ch} &= \frac{\partial^2}{\partial l\,\partial m} + \frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big) - \frac{1}{(l-m)^2}\left(1 - \left(\gamma - \frac{1}{2}\right)^2\right) .
\end{aligned}$$

The direct calculations show that, if function u solves the equation $\mathcal{S}_{ch}u = 0$, then the function $v = (l-m)u$ solves the equation $\mathcal{S}^*_{ch}v = 0$, and vice versa. Then Lemma 1.2 completes the proof. Lemma is proven. □

2 Proof of Theorem 0.1

Next we use Riemann function $R(l,m;a,b)$ and function $E(x,t;x_0,t_0)$ defined by (0.15) to complete the proof of Theorem 0.1, which gives the fundamental solution with a support in the forward cone $D_+(x_0,t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, and the fundamental solution with a support in the backward cone $D_-(x_0,t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, defined by (0.14) with plus and minus, respectively.

We present a proof for $\mathcal{E}_+(x,t;0,b)$ since for $\mathcal{E}_-(x,t;0,b)$ it is similar. First, we note that the operator $\mathcal{S}$ is formally self-adjoint, $\mathcal{S} = \mathcal{S}^*$. We must show that

$$< \mathcal{E}_+, \mathcal{S}\varphi >= \varphi(0,b)\,, \qquad \text{for every} \quad \varphi \in C_0^\infty(\mathbb{R}^2)\,.$$

Since $E(x,t;0,b)$ is locally integrable in $\mathbb{R}^2$, this is equivalent to showing that

$$\iint_{\mathbb{R}^2} \mathcal{E}_+(x,t;0,b)\mathcal{S}\varphi(x,t)\,dx\,dt = \varphi(0,b), \quad \text{for every} \quad \varphi \in C_0^\infty(\mathbb{R}^2). \tag{2.1}$$

In the mean time $D(x,t)/D(l,m) = (l-m)^{-1}$ is the Jacobian of the transformation (1.1). Hence the integral in the left-hand side of (2.1) is equal to

$$\begin{aligned}
&\iint_{\mathbb{R}^2} \mathcal{E}_+(x,t;0,b)\mathcal{S}\varphi(x,t)\,dx\,dt = \int_b^\infty dt \int_{e^{-t}-e^{-b}}^{e^{-b}-e^{-t}} E(x,t;0,b)\mathcal{S}\varphi(x,t)\,dx\\
&= -\int_{-e^{-b}}^{\infty}\int_{-\infty}^{e^{-b}} R(l,m;e^{-b},-e^{-b})\,dl\,dm\left\{\frac{\partial^2}{\partial l\,\partial m} - \frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial m}\Big) - \frac{1}{(l-m)^2}M^2\right\}\varphi.
\end{aligned}$$

We consider the first term of the right hand side, and integrate it by parts

$$\begin{aligned}
&\int_{-e^{-b}}^{\infty} dm \int_{-\infty}^{e^{-b}} dl\ R(l,m;e^{-b},-e^{-b})\frac{\partial^2}{\partial l\,\partial m}\varphi\\
&= \int_{-e^{-b}}^{\infty} R(e^{-b},m;e^{-b},-e^{-b})\frac{\partial\varphi}{\partial m}\Big|_{l=e^{-b}} dm - \left[-\int_{-\infty}^{e^{-b}} dl\left(\frac{\partial}{\partial l}R(l,-e^{-b};e^{-b},-e^{-b})\right)\varphi\Big|_{m=-e^{-b}}\right.\\
&\qquad\left. - \int_{-e^{-b}}^{\infty} dm \int_{-\infty}^{e^{-b}} dl\left(\frac{\partial^2}{\partial l\,\partial m}R(l,m;e^{-b},-e^{-b})\right)\varphi\right]\\
&= -\varphi(e^{-b},-e^{-b}) - \int_{-e^{-b}}^{\infty} dm\left(\frac{\partial}{\partial m}R(e^{-b},m;e^{-b},-e^{-b})\right)\varphi(e^{-b},m)\\
&\quad + \int_{-\infty}^{e^{-b}} dl\left(\frac{\partial}{\partial l}R(l,-e^{-b};e^{-b},-e^{-b})\right)\varphi(l,-e^{-b}) + \int_{-e^{-b}}^{\infty} dm\int_{-\infty}^{e^{-b}} dl\left(\frac{\partial^2}{\partial l\,\partial m}R(l,m;e^{-b},-e^{-b})\right)\varphi.
\end{aligned}$$

Then, for the second term we obtain

$$\begin{aligned}
& -\int_{-e^{-b}}^{\infty} dm \int_{-\infty}^{e^{-b}} dl\, R(l,m;e^{-b},-e^{-b}) \frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l}-\frac{\partial}{\partial m}\Big)\varphi \\
= & -\int_{-\infty}^{e^{-b}} R(l,-e^{-b};e^{-b},-e^{-b})\frac{1}{2(l+e^{-b})}\varphi(l,-e^{-b})\,dl \\
& -\int_{-e^{-b}}^{\infty} R(e^{-b},m;e^{-b},-e^{-b})\frac{1}{2(e^{-b}-m)}\varphi(e^{-b},m)\,dm \\
& -\int_{-e^{-b}}^{\infty} dm \int_{-\infty}^{e^{-b}} dl\, \frac{1}{(l-m)^2} R(l,m;e^{-b},-e^{-b})\varphi(l,m) \\
& +\int_{-e^{-b}}^{\infty} dm \int_{-\infty}^{e^{-b}} dl\, \Big[\frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l}-\frac{\partial}{\partial m}\Big)R(l,m;e^{-b},-e^{-b})\Big]\varphi(l,m)\,.
\end{aligned}$$

Using properties of the Riemann function we derive

$$\begin{aligned}
& \int_{-e^{-b}}^{\infty} dm \int_{-\infty}^{e^{-b}} R(l,m;e^{-b},-e^{-b})\,dl \left\{\frac{\partial^2}{\partial l\,\partial m}-\frac{1}{2(l-m)}\Big(\frac{\partial}{\partial l}-\frac{\partial}{\partial m}\Big)-\frac{1}{(l-m)^2}M^2\right\}\varphi \\
= & -\varphi(e^{-b},-e^{-b}) - \int_{-e^{-b}}^{\infty}\left(\frac{\partial}{\partial m}R(e^{-b},m;e^{-b},-e^{-b})\right)\varphi(e^{-b},m)\,dm \\
& +\int_{-\infty}^{e^{-b}}\left(\frac{\partial}{\partial l}R(l,-e^{-b};e^{-b},-e^{-b})\right)\varphi(l,-e^{-b})\,dl \\
& -\int_{-\infty}^{e^{-b}} R(l,-e^{-b};e^{-b},-e^{-b})\frac{1}{2(l+e^{-b})}\varphi(l,-e^{-b})\,dl \\
& -\int_{-e^{-b}}^{\infty} R(e^{-b},m;e^{-b},-e^{-b})\frac{1}{2(e^{-b}-m)}\varphi(e^{-b},m)\,dm \\
= & -\varphi(e^{-b},-e^{-b})\,.
\end{aligned}$$

Theorem is proven. □

3 Application to the Cauchy Problem: Source Term and $n=1$

Consider now the Cauchy problem for the equation (0.18) with vanishing initial data (0.19). For every $(x,t)\in D_+(0,b)$ one has $e^{-t}-e^{-b}\le x\le e^{-b}-e^{-t}$, so that

$$E(x,t;0,b) \quad = \quad (4e^{-b-t})^{iM}\Big((e^{-t}+e^{-b})^2-x^2\Big)^{-\frac{1}{2}-iM}F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-b}-e^{-t})^2-x^2}{(e^{-b}+e^{-t})^2-x^2}\Big).$$

The coefficient of the equation (0.9) is independent of x, therefore $\mathcal{E}_+(x,t;\,y,b)=\mathcal{E}_+(x-y,t;0,b)$. Using the fundamental solution from Theorem 0.1 one can write the convolution

$$u(x,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{E}_+(x,t;y,b)f(y,b)\,db\,dy=\int_0^t db\int_{-\infty}^{\infty}\mathcal{E}_+(x-y,t;0,b)f(y,b)\,dy, \tag{3.1}$$

which is well-defined since supp$f\subset\{t\ge 0\}$. Then according to the definition of the distribution $\mathcal{E}_+$ we obtain the statement of the Theorem 0.3. Thus, Theorem 0.3 is proven.

Remark 3.1 *The argument of the hypergeometric function is nonnegative and bounded,*

$$0\le\frac{(e^{-b}-e^{-t})^2-z^2}{(e^{-b}+e^{-t})^2-z^2}<1 \quad \textit{for all} \quad b\in(0,t),\ z\in(e^{-t}-e^{-b},e^{-b}-e^{-t})\,.$$

The following corollary is a manifestation of the time-speed transformation principle introduced in [29]. It implies the existence of an operator transforming the solutions of the Cauchy problem for the string equation to the solutions of the Cauchy problem for the inhomogeneous equation with time-dependent speed of propagation. One may think of this transformation as a "two-stage" Duhamel's principal, but unlike the last one, it reduces the equation with the time-dependent speed of propagation to the one with the speed of propagation independent of time.

Corollary 3.2 *The solution $u = u(x,t)$ of the Cauchy problem (0.18)-(0.19) can be represented by (0.24), where the functions $v(x,t;\tau) := \frac{1}{2}(f(x+t,\tau)+f(x-t,\tau))$, $\tau \in [0,\infty)$, form a one-parameter family of solutions to the Cauchy problem for the string equation, that is, $v_{tt} - v_{xx} = 0$, $v(x,0;\tau) = f(x,\tau)$, $v_t(x,0;\tau) = 0$.*

Proof. From the convolution (3.1) we derive

$$\begin{aligned} u(x,t) &= \int_0^t db \int_{-(e^{-b}-e^{-t})}^{e^{-b}-e^{-t}} dz\, f(z+x,b)(4e^{-b-t})^{iM}\Big((e^{-t}+e^{-b})^2 - z^2\Big)^{-\frac{1}{2}-iM} \\ &\quad \times F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-b}-e^{-t})^2-z^2}{(e^{-b}+e^{-t})^2-z^2}\Big) \\ &= 2\int_0^t db \int_0^{e^{-b}-e^{-t}} dz\, \frac{1}{2}\{f(x+z,b)+f(x-z,b)\}(4e^{-b-t})^{iM}\Big((e^{-t}+e^{-b})^2 - z^2\Big)^{-\frac{1}{2}-iM} \\ &\quad \times F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-b}-e^{-t})^2-z^2}{(e^{-b}+e^{-t})^2-z^2}\Big)\,. \end{aligned}$$

Corollary is proven. □

Some Properties of the Function $\mathbf{E(x,t;y,b)}$.

In this section we collect some elementary auxiliary formulas in order to make the proofs of main theorems more transparent.

Proposition 3.3 *Let $E(x,t;x_0,t_0)$ be function defined by (0.15). One has*

$$E(x,t;y,b) = E(y,b;x,t), \tag{3.2}$$

$$E(x,t;y,b) = E(x-y,t;0,b) \quad , \quad E(x,t;0,b) = E(-x,t;0,b), \tag{3.3}$$

$$E(x,t;0,-\ln(x+e^{-t})) = \frac{1}{2}\frac{1}{\sqrt{e^{-t}}\sqrt{x+e^{-t}}}, \tag{3.4}$$

$$\frac{\partial}{\partial b}\Big(e^{-b}E(e^{-b}-e^{-t},t;0,b)\Big) = -\frac{1}{4}e^{t/2}e^{-b/2}, \tag{3.5}$$

$$\frac{\partial}{\partial b}\Big(be^{-b}E(-e^{-b}+e^{-t},t;0,b)\Big) = \frac{\partial}{\partial b}\Big(be^{-b}E(e^{-b}-e^{-t},t;0,b)\Big) = \frac{1}{4}e^{t/2}e^{-b/2}(2-b), \tag{3.6}$$

$$\lim_{y\to x+e^{-b}-e^{-t}} \frac{\partial}{\partial x}E(x-y,t;0,b) = -\frac{1}{16}(1+4M^2)e^{t/2}e^{b/2}(e^t-e^b), \tag{3.7}$$

$$\lim_{y\to x-e^{-b}+e^{-t}} \frac{\partial}{\partial x}E(x-y,t;0,b) = \frac{1}{16}(1+4M^2)e^{t/2}e^{b/2}(e^t-e^b), \tag{3.8}$$

$$\left[\frac{\partial}{\partial b}E(x,t;0,b)\right]_{b=-\ln(x+e^{-t})} = \frac{1}{16}e^t\frac{4+e^tx(1+4M^2)}{\sqrt{1+e^tx}}, \tag{3.9}$$

$$\begin{aligned} \left[\frac{\partial}{\partial b}E(x,t;0,b)\right]_{b=0} &= -(4e^{-t})^{iM}\big((e^{-t}+1)^2-x^2\big)^{-iM}\frac{1}{[(1-e^{-t})^2-x^2]\sqrt{(e^{-t}+1)^2-x^2}} \\ &\quad \times\Bigg[\big(e^{-t}-1-iM(e^{-2t}-1-x^2)\big)F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(1-e^{-t})^2-x^2}{(1+e^{-t})^2-x^2}\Big) \\ &\quad +\big(1-e^{-2t}+x^2\big)\Big(\frac{1}{2}-iM\Big)F\Big(-\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(1-e^{-t})^2-x^2}{(1+e^{-t})^2-x^2}\Big)\Bigg]. \end{aligned} \tag{3.10}$$

Proof. The properties (3.2),(3.3), and (3.4) are evident. To prove (3.5) and (3.6) we write

$$E(e^{-b}-e^{-t},t;0,b)=(4e^{-b-t})^{-\frac{1}{2}},$$

that implies both (3.5) and (3.6). To prove (3.7) we denote

$$z:=\frac{(e^{-b}-e^{-t})^2-(x-y)^2}{(e^{-b}+e^{-t})^2-(x-y)^2}\,,$$

so that,

$$\frac{\partial z}{\partial x} \;=\; -2(x-y)\frac{4e^{-b}e^{-t}}{[(e^{-b}+e^{-t})^2-(x-y)^2]^2}\,, \qquad \frac{\partial z}{\partial b}=-\frac{4e^{-b}e^{-t}\left(-e^{-2t}+e^{-2b}+(x-y)^2\right)}{\left[(e^{-b}+e^{-t})^2-(x-y)^2\right]^2}\,.$$

Then we obtain

$$\begin{aligned}\frac{\partial}{\partial x}E(x-y,t;0,b) \;=\;& (4e^{-b-t})^{iM}\Bigg[(-2(x-y))(-\frac{1}{2}-iM)\Big((e^{-t}+e^{-b})^2-(x-y)^2\Big)^{-\frac{3}{2}-iM}\\ &\times F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;z\Big)\\ &+\Big((e^{-t}+e^{-b})^2-(x-y)^2\Big)^{-\frac{1}{2}-iM}\Big(\frac{\partial z}{\partial x}\Big)F_z\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;z\Big)\Bigg]. \end{aligned}\qquad (3.11)$$

It is easily seen that

$$\lim_{y\to x+e^{-b}-e^{-t}} z=0, \qquad \lim_{y\to x+e^{-b}-e^{-t}}\frac{\partial z}{\partial x}=\frac{1}{2}(e^t-e^b),$$

while according to (20) [2, Sec.2.8 v.1], we have

$$\lim_{y\to x+e^{-b}-e^{-t}}\partial_z F\left(\frac{1}{2}+iM,\frac{1}{2}+iM;1;z\right) \;=\; \frac{\Gamma^2(\frac{3}{2}+iM)}{\Gamma^2(\frac{1}{2}+iM)}=\left(\frac{1}{2}+iM\right)^2. \qquad (3.12)$$

Consequently, from (3.11) we obtain

$$\begin{aligned}&\lim_{y\to x+e^{-b}-e^{-t}}\frac{\partial}{\partial x}E(x-y,t;0,b)\\ =\;& (4e^{-b-t})^{iM}\left[2(e^{-b}-e^{-t})\left(-\frac{1}{2}-iM\right)\left(4e^{-t}e^{-b}\right)^{-\frac{3}{2}-iM}+\left(4e^{-t}e^{-b}\right)^{-\frac{1}{2}-iM}\frac{1}{2}(e^t-e^b)\left(\frac{1}{2}+iM\right)^2\right]\\ =\;& -e^{\frac{t}{2}}e^{\frac{b}{2}}\frac{1}{16}(e^t-e^b)\Big[1+4M^2\Big]\,.\end{aligned}$$

The proof of (3.8) is similar. To prove (3.9) we write

$$\begin{aligned}&\frac{\partial}{\partial b}E(x,t;0,b)\\ =\;& (-iM)(4e^{-b-t})^{iM}\Big((e^{-t}+e^{-b})^2-x^2\Big)^{-\frac{1}{2}-iM}F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-b}-e^{-t})^2-x^2}{(e^{-b}+e^{-t})^2-x^2}\Big)\\ &-\Big(\frac{1}{2}+iM\Big)(4e^{-b-t})^{iM}(-e^{-b})2(e^{-t}+e^{-b})\Big((e^{-t}+e^{-b})^2-x^2\Big)^{-\frac{3}{2}-iM}\\ &\times F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-b}-e^{-t})^2-x^2}{(e^{-b}+e^{-t})^2-x^2}\Big)\\ &+(4e^{-b-t})^{iM}\Big((e^{-t}+e^{-b})^2-x^2\Big)^{-\frac{1}{2}-iM}\frac{\partial}{\partial b}\Bigg(F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-b}-e^{-t})^2-x^2}{(e^{-b}+e^{-t})^2-x^2}\Big)\Bigg).\end{aligned}\qquad (3.13)$$

On the other hand (3.12) and (3.13) imply

$$
\begin{aligned}
&\Big[\frac{\partial}{\partial b}E(x,t;0,b)\Big]_{b=-\ln(x+e^{-t})}\\
= \;& (4e^{-b-t})^{iM}\Big(4e^{-b-t}\Big)^{-\frac{1}{2}-iM}\Bigg[(-iM)+2\Big(\frac{1}{2}+iM\Big)e^{-b}(e^{-t}+e^{-b})\Big(4e^{-b-t}\Big)^{-1}\\
&\qquad +\frac{\partial}{\partial b}\Bigg(F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-b}-e^{-t})^2-x^2}{(e^{-b}+e^{-t})^2-x^2}\Big)\Bigg)\Bigg]_{b=-\ln(x+e^{-t})}\\
= \;& \frac{1}{2}e^{\frac{1}{2}b+\frac{1}{2}t}\Bigg[(-iM)+\frac{1}{2}\Big(\frac{1}{2}+iM\Big)(e^{-t}+e^{-b})e^{t}-\frac{4e^{-b}e^{-t}\left(e^{-2b}-e^{-2t}+x^2\right)}{[(e^{-b}+e^{-t})^2-x^2]^2}\Big(\frac{1}{2}+iM\Big)^2\Bigg]_{b=-\ln(x+e^{-t})}\\
= \;& \frac{1}{2}\frac{1}{\sqrt{1+xe^t}}e^t\Bigg[-iM+\frac{1}{2}\Big(\frac{1}{2}+iM\Big)(2+xe^t)-\frac{xe^t}{2}\Big(\frac{1}{2}+iM\Big)^2\Bigg]\\
= \;& \frac{1}{2}\frac{1}{\sqrt{1+xe^t}}e^t\frac{4+xe^t(1+4M^2)}{8}\,.
\end{aligned}
$$

Thus, (3.9) is proven. To prove (3.10) we appeal to (23)[2, v.1, Sec.2.8], that reads with $a=b=\frac{1}{2}+iM$, $c=1$,

$$
\begin{aligned}
F_z\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;z\Big) \;=\;& \frac{1}{z(1-z)}\Bigg\{\Big[-\Big(\frac{1}{2}-iM\Big)+\Big(\frac{1}{2}+iM\Big)z\Big]F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;z\Big)\\
&\qquad +\Big(\frac{1}{2}-iM\Big)F\Big(-\frac{1}{2}+iM,\frac{1}{2}+iM;1;z\Big)\Bigg\}.
\end{aligned}
$$

Then we plug the last relation in (3.13) and obtain

$$
\begin{aligned}
&\Big[\frac{\partial}{\partial b}E(x,t;0,b)\Big]_{b=0}\\
= \;& -iM(4e^{-t})^{iM}\Big((e^{-t}+1)^2-x^2\Big)^{-\frac{1}{2}-iM}F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\zeta_0\Big)\\
& -\Big(\frac{1}{2}+iM\Big)(4e^{-t})^{iM}(-1)2(e^{-t}+1)\Big((e^{-t}+1)^2-x^2\Big)^{-\frac{3}{2}-iM}F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\zeta_0\Big)\\
& +(4e^{-t})^{iM}\Big((e^{-t}+1)^2-x^2\Big)^{-\frac{1}{2}-iM}\Bigg[\frac{\partial}{\partial b}\Bigg(F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\zeta\Big)\Bigg)\Bigg]_{b=0}\,,
\end{aligned}
$$

where we have denoted

$$
\zeta := \frac{(e^{-b}-e^{-t})^2-x^2}{(e^{-b}+e^{-t})^2-x^2}\,,\qquad \zeta_0 := \frac{(1-e^{-t})^2-x^2}{(1+e^{-t})^2-x^2}\,,\qquad \zeta_0(1-\zeta_0)=\frac{4e^{-t}[(1-e^{-t})^2-x^2]}{[(1+e^{-t})^2-x^2]^2}\,,
$$

with

$$
\frac{\partial}{\partial b}\zeta = -\frac{4e^{-b}e^{-t}\left(e^{-2b}-e^{-2t}+x^2\right)}{[(e^{-b}+e^{-t})^2-x^2]^2}\,,\qquad \frac{\partial\zeta}{\partial b}\Big|_{b=0} = -\frac{4e^{-t}\left(1-e^{-2t}+x^2\right)}{[(1+e^{-t})^2-x^2]^2}\,.
$$

Hence, due to (20) [2, v.1, Sec.2.8], we obtain

$$
\begin{aligned}
&\Bigg[\frac{\partial}{\partial b}F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-b}-e^{-t})^2-x^2}{(e^{-b}+e^{-t})^2-x^2}\Big)\Bigg]_{b=0}\\
= \;& -\frac{1-e^{-2t}+x^2}{(1-e^{-t})^2-x^2}\Bigg\{\Big[-\Big(\frac{1}{2}-iM\Big)+\Big(\frac{1}{2}+iM\Big)\zeta_0\Big]F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\zeta_0\Big)\\
&\qquad +\Big(\frac{1}{2}-iM\Big)F\Big(-\frac{1}{2}+iM,\frac{1}{2}+iM;1;\zeta_0\Big)\Bigg\}.
\end{aligned}
$$

Hence,

$$\begin{aligned}
&\left[\frac{\partial}{\partial b}E(x,t;0,b)\right]_{b=0}\\
= \;& -iM(4e^{-t})^{iM}\Big((e^{-t}+1)^2-x^2\Big)^{-\frac{1}{2}-iM}F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\zeta_0\Big)\\
&-\Big(\frac{1}{2}+iM\Big)(4e^{-t})^{iM}(-1)2(e^{-t}+1)\Big((e^{-t}+1)^2-x^2\Big)^{-\frac{3}{2}-iM}F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\zeta_0\Big)\\
&+(4e^{-t})^{iM}\Big((e^{-t}+1)^2-x^2\Big)^{-\frac{1}{2}-iM}\\
&\times\Bigg[-\frac{1-e^{-2t}+x^2}{(1-e^{-t})^2-x^2}\Bigg\{\Big[-\Big(\frac{1}{2}-iM\Big)+\Big(\frac{1}{2}+iM\Big)\zeta_0\Big]F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\zeta_0\Big)\\
&+\Big(\frac{1}{2}-iM\Big)F\Big(-\frac{1}{2}+iM,\frac{1}{2}+iM;1;\zeta_0\Big)\Bigg\}\Bigg]\\
= \;& (4e^{-t})^{iM}\Big((e^{-t}+1)^2-x^2\Big)^{-\frac{1}{2}-iM}\Bigg[F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\zeta_0\Big)\Big\{-iM+(1+2iM)\frac{e^{-t}+1}{(e^{-t}+1)^2-x^2}\\
&-\frac{1-e^{-2t}+x^2}{(1-e^{-t})^2-x^2}\Big[-\Big(\frac{1}{2}-iM\Big)+\Big(\frac{1}{2}+iM\Big)\frac{(1-e^{-t})^2-x^2}{(1+e^{-t})^2-x^2}\Big]\Big\}\\
&-\frac{1-e^{-2t}+x^2}{(1-e^{-t})^2-x^2}\Big(\frac{1}{2}-iM\Big)F\Big(-\frac{1}{2}+iM,\frac{1}{2}+iM;1;\zeta_0\Big)\Bigg].
\end{aligned}$$

Finally,

$$\begin{aligned}
\left[\frac{\partial}{\partial b}E(x,t;0,b)\right]_{b=0} \;=\;& (4e^{-t})^{iM}\big((e^{-t}+1)^2-x^2\big)^{-\frac{1}{2}-iM}\\
&\times\Bigg[\frac{e^{t}-iM+e^{2t}\big(iM(1+x^2)-1\big)}{e^{t}\big(2+e^{t}(x^2-1)\big)-1}F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\zeta_0\Big)\\
&\quad-\frac{1-e^{-2t}+x^2}{(1-e^{-t})^2-x^2}\Big(\frac{1}{2}-iM\Big)F\Big(-\frac{1}{2}+iM,\frac{1}{2}+iM;1;\zeta_0\Big)\Bigg].
\end{aligned}$$

The formula (3.10) and, consequently, the proposition are proven. □

4 The Cauchy Problem: Second Datum and $n=1$

In this section we prove Theorem 0.4 in the case of $\varphi_0(x)=0$. More precisely, we have to prove that the solution $u(x,t)$ of the Cauchy problem (0.20) with $\varphi_0(x)=0$ and $\varphi_1(x)=\varphi(x)$ can be represented as follows

$$u(x,t)=\int_0^{1-e^{-t}}\Big[\varphi(x+z)+\varphi(x-z)\Big]K_1(z,t)dz=\int_0^1\Big[\varphi(x+\phi(t)s)+\varphi(x-\phi(t)s)\Big]K_1(\phi(t)s,t)\phi(t)ds, \quad (4.1)$$

where $\phi(t)=1-e^{-t}$. The proof of the theorem is splitted into several steps.

Proposition 4.1 *The solution $u=u(x,t)$ of the Cauchy problem (0.20) with $\varphi_0(x)=0$ and $\varphi_1(x)=\varphi(x)$ can be represented as follows*

$$\begin{aligned}
u(x,t) \;=\;& \int_0^t db\,\Big[\frac{1}{4}e^{t/2}e^{-b/2}(2-b)+\frac{1}{16}be^{-3b/2}e^{t/2}(e^{b}-e^{t})(1+4M^2)\Big]\\
&\qquad\times\Big[\varphi(x+e^{-b}-e^{-t})+\varphi(x-e^{-b}+e^{-t})\Big]\\
&+\int_0^t db\int_{x-(e^{-b}-e^{-t})}^{x+e^{-b}-e^{-t}}dy\,\varphi(y)b\Big[e^{-2b}\,\Big(\frac{\partial}{\partial y}\Big)^2E(x-y,t;0,b)-M^2E(x-y,t;0,b)\Big]. \quad (4.2)
\end{aligned}$$

Proof. We look for the solution $u = u(x,t)$ of the form $u(x,t) = w(x,t) + t\varphi(x)$. Then (0.20) implies

$$w_{tt} - e^{-2t}w_{xx} + M^2 w = te^{-2t}\varphi^{(2)}(x) - M^2 t\varphi(x), \qquad w(x,0) = 0, \quad w_t(x,0) = 0\,.$$

We set $f(x,t) = te^{-2t}\varphi^{(2)}(x) - M^2 t\varphi(x)$ and due to Theorem 0.3 obtain

$$w(x,t) = \widetilde{w(x,t)} - M^2 \int_0^t b\,db \int_{x-(e^{-b}-e^{-t})}^{x+e^{-b}-e^{-t}} dy\,\varphi(y)E(x-y,t;0,b),$$

where

$$\widetilde{w(x,t)} \quad := \quad \int_0^t be^{-2b}\,db \int_{x-(e^{-b}-e^{-t})}^{x+e^{-b}-e^{-t}} dy\,\varphi^{(2)}(y)E(x-y,t;0,b)\,.$$

Then we integrate by parts:

$$\widetilde{w(x,t)} = \int_0^t be^{-2b}\,db\Big[\varphi^{(1)}(x+e^{-b}-e^{-t})E(-e^{-b}+e^{-t},t;0,b) - \varphi^{(1)}(x-e^{-b}+e^{-t})E(e^{-b}-e^{-t},t;0,b)\Big]$$
$$- \int_0^t be^{-2b}\,db \int_{x-(e^{-b}-e^{-t})}^{x+e^{-b}-e^{-t}} dy\,\varphi^{(1)}(y)\frac{\partial}{\partial y}E(x-y,t;0,b)\,.$$

But

$$\varphi^{(1)}(x+e^{-b}-e^{-t}) = -e^{b}\frac{\partial}{\partial b}\varphi(x+e^{-b}-e^{-t}) \quad \text{and} \quad \varphi^{(1)}(x-e^{-b}+e^{-t}) = e^{b}\frac{\partial}{\partial b}\varphi(x-e^{-b}+e^{-t})\,.$$

Then, $E(e^{-b}-e^{-t},t;0,b) = E(-e^{-b}+e^{-t},t;0,b)$ due to (3.3), and we obtain

$$\widetilde{w(x,t)} \quad = \quad \int_0^t be^{-2b}\,db\Big[-e^{b}\frac{\partial}{\partial b}\varphi(x+e^{-b}-e^{-t}) - e^{b}\frac{\partial}{\partial b}\varphi(x-e^{-b}+e^{-t})\Big]E(e^{-b}-e^{-t},t;0,b)$$
$$- \int_0^t be^{-2b}\,db \int_{x-(e^{-b}-e^{-t})}^{x+e^{-b}-e^{-t}} dy\,\varphi^{(1)}(y)\frac{\partial}{\partial y}E(x-y,t;0,b)\,.$$

One more integration by parts leads to

$$\begin{aligned}
\widetilde{w(x,t)} \quad &= \quad -2te^{-t}\varphi(x)E(0,t;0,t)\\
&\quad + \int_0^t db\Big(\varphi(x+e^{-b}-e^{-t}) + \varphi(x-e^{-b}+e^{-t})\Big)\frac{\partial}{\partial b}\Big(be^{-b}E(e^{-b}-e^{-t},t;0,b)\Big)\\
&\quad - \int_0^t be^{-2b}\,db \int_{x-(e^{-b}-e^{-t})}^{x+e^{-b}-e^{-t}} dy\,\varphi^{(1)}(y)\frac{\partial}{\partial y}E(x-y,t;0,b)\,.
\end{aligned}$$

Since $E(0,t;0,t) = e^t/2$ we use (3.6) of Proposition 3.3 to derive the next representation

$$\begin{aligned}
\widetilde{w(x,t)} + t\varphi(x) \quad &= \quad \int_0^t \frac{1}{4}e^{t/2}e^{-b/2}(2-b)\Big(\varphi(x+e^{-b}-e^{-t}) + \varphi(x-e^{-b}+e^{-t})\Big)\,db\\
&\quad - \int_0^t be^{-2b}\,db \int_{x-(e^{-b}-e^{-t})}^{x+e^{-b}-e^{-t}} dy\,\varphi^{(1)}(y)\frac{\partial}{\partial y}E(x-y,t;0,b)\,.
\end{aligned}$$

The integration by parts in the second term leads to

$$\begin{aligned}
\widetilde{w(x,t)} + t\varphi(x) \quad &= \quad \int_0^t \frac{1}{4}e^{t/2}e^{-b/2}(2-b)\Big[\varphi(x+e^{-b}-e^{-t}) + \varphi(x-e^{-b}+e^{-t})\Big]\,db\\
&\quad - \int_0^t be^{-2b}\,db\,\varphi(x+e^{-b}-e^{-t})\left[\frac{\partial}{\partial y}E(x-y,t;0,b)\right]_{y=x+e^{-b}-e^{-t}}\\
&\quad + \int_0^t be^{-2b}\,db\,\varphi(x-e^{-b}+e^{-t})\left[\frac{\partial}{\partial y}E(x-y,t;0,b)\right]_{y=x-e^{-b}+e^{-t}}\\
&\quad + \int_0^t be^{-2b}\,db \int_{x-(e^{-b}-e^{-t})}^{x+e^{-b}-e^{-t}} dy\,\varphi(y)\left(\frac{\partial}{\partial y}\right)^2 E(x-y,t;0,b)\,.
\end{aligned}$$

The application of (3.7) and (3.8) of Proposition 3.3 and $\frac{\partial}{\partial y}E(x-y,t;0,b) = -\frac{\partial}{\partial x}E(x-y,t;0,b)$ imply

$$
\begin{aligned}
&\widetilde{w(x,t)} + t\varphi(x) \\
= \ & \int_0^t \Big[\frac{1}{4}e^{t/2}e^{-b/2}(2-b) + \frac{1}{16}be^{-3b/2}e^{t/2}(e^b-e^t)(1+4M^2)\Big]\Big[\varphi(x+e^{-b}-e^{-t}) + \varphi(x-e^{-b}+e^{-t})\Big]\, db \\
& + \int_0^t be^{-2b}\, db \int_{x-e^{-b}+e^{-t}}^{x+e^{-b}-e^{-t}} dy\, \varphi(y) \left(\frac{\partial}{\partial y}\right)^2 E(x-y,t;0,b)\,.
\end{aligned}
$$

Thus, for the function $u(x,t) = w(x,t) + t\varphi(x)$ we have obtained

$$
\begin{aligned}
&u(x,t) \\
= \ & \int_0^t \Big[\frac{1}{4}e^{t/2}e^{-b/2}(2-b) + \frac{1}{16}be^{-3b/2}e^{t/2}(e^b-e^t)(1+4M^2)\Big]\Big[\varphi(x+e^{-b}-e^{-t}) + \varphi(x-e^{-b}+e^{-t})\Big]\, db \\
& + \int_0^t be^{-2b}\, db \int_{x-e^{-b}+e^{-t}}^{x+e^{-b}-e^{-t}} dy\, \varphi(y) \left(\frac{\partial}{\partial y}\right)^2 E(x-y,t;0,b) \\
& - M^2 \int_0^t b\, db \int_{x-e^{-b}+e^{-t}}^{x+e^{-b}-e^{-t}} dy\, \varphi(y) E(x-y,t;0,b)\ . \qquad (4.3)
\end{aligned}
$$

The proposition is proven. □

Corollary 4.2 *The solution $u = u(x,t)$ of the Cauchy problem (0.20) with $\varphi_0(x) = 0$ and $\varphi_1(x) = \varphi(x)$ can be represented as follows*

$$
\begin{aligned}
&u(x,t) \\
= \ & \int_0^t \Big[\frac{1}{4}e^{t/2}e^{-b/2}(2-b) + \frac{1}{16}be^{-3b/2}e^{t/2}(e^b-e^t)(1+4M^2)\Big]\Big[\varphi(x+e^{-b}-e^{-t}) + \varphi(x-e^{-b}+e^{-t})\Big]\, db \\
& + \int_0^t db \int_0^{e^{-b}-e^{-t}} dz\, \Big[\varphi(x-z) + \varphi(x+z)\Big] b \left[e^{-2b}\left(\frac{\partial}{\partial z}\right)^2 E(z,t;0,b) - M^2 E(z,t;0,b)\right], \qquad (4.4)
\end{aligned}
$$

as well as by (4.1), where

$$
\begin{aligned}
K_1(z,t) \ = \ & \Big[\frac{1}{4}e^{t/2}\big(2 + \ln(z+e^{-t})\big) + \frac{1}{16}(1+4M^2)e^{3t/2} z \ln(z+e^{-t})\Big]\frac{1}{\sqrt{z+e^{-t}}} \\
& + \int_0^{-\ln(z+e^{-t})} b\left[e^{-2b}\left(\frac{\partial}{\partial z}\right)^2 E(z,t;0,b) - M^2 E(z,t;0,b)\right] db\,. \qquad (4.5)
\end{aligned}
$$

Proof of Corollary. By means of the statement (4.2) of Proposition 4.1 and (3.3) we obtain

$$
\begin{aligned}
u(x,t) \ = \ & \int_0^t db\, \Big[\frac{1}{4}e^{t/2}e^{-b/2}(2-b) + \frac{1}{16}be^{-3b/2}e^{t/2}(e^b-e^t)(1+4M^2)\Big] \\
& \times\Big[\varphi(x+e^{-b}-e^{-t}) + \varphi(x-e^{-b}+e^{-t})\Big] \\
& + \int_0^t db \int_0^{-(e^{-b}-e^{-t})} (-1) dz\, \varphi(x-z) \left[be^{-2b}\left(\frac{\partial}{\partial z}\right)^2 E(z,t;0,b) - M^2 b E(z,t;0,b)\right] \\
& + \int_0^t db \int_{-(e^{-b}-e^{-t})}^0 dz\, \varphi(x+z) \left[be^{-2b}\left(\frac{\partial}{\partial z}\right)^2 E(z,t;0,b) - M^2 b E(z,t;0,b)\right].
\end{aligned}
$$

To prove (4.1) with $K_1(z,t)$ defined by (4.5) we apply (3.3) and write

$$
u(x,t) \ = \ \int_0^t db\, \Big[\frac{1}{4}e^{t/2}e^{-b/2}(2-b) + \frac{1}{16}be^{-3b/2}e^{t/2}(e^b-e^t)(1+4M^2)\Big]
$$

$$
\times\Big[\varphi(x+e^{-b}-e^{-t})+\varphi(x-e^{-b}+e^{-t})\Big]
$$

$$
+\int_0^t db\int_0^{e^{-b}-e^{-t}} dz\,\Big[\varphi(x+z)+\varphi(x-z)\Big]\left[be^{-2b}\left(\frac{\partial}{\partial z}\right)^2 E(z,t;0,b)-M^2bE(z,t;0,b)\right].
$$

Next we make change $z=e^{-b}-e^{-t}$, $\;dz=-e^{-b}db$, $db=-(z+e^{-t})^{-1}dz$, and $b=-\ln(z+e^{-t})$ in the first integral:

$$
\begin{aligned}
&\int_0^t\Big[\frac{1}{4}e^{t/2}e^{-b/2}(2-b)+\frac{1}{16}be^{-3b/2}e^{t/2}(e^b-e^t)(1+4M^2)\Big]\Big[\varphi(x+e^{-b}-e^{-t})+\varphi(x-e^{-b}+e^{-t})\Big]db\\
=\;&\int_0^{1-e^{-t}}\Big[\frac{1}{4}e^{t/2}\big(2+\ln(z+e^{-t})\big)+\frac{1}{16}(1+4M^2)e^{3t/2}z\ln(z+e^{-t})\Big]\frac{1}{\sqrt{z+e^{-t}}}\\
&\qquad\times\Big[\varphi(x+z)+\varphi(x-z)\Big]dz\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
u(x,t)\;=\;&\int_0^{1-e^{-t}}\Big[\frac{1}{4}e^{t/2}\big(2+\ln(z+e^{-t})\big)+\frac{1}{16}(1+4M^2)e^{3t/2}z\ln(z+e^{-t})\Big]\frac{1}{\sqrt{z+e^{-t}}}\\
&\qquad\times\Big[\varphi(x+z)+\varphi(x-z)\Big]dz\\
&+\int_0^t db\int_0^{e^{-b}-e^{-t}} dz\,\Big[\varphi(x+z)+\varphi(x-z)\Big]b\left[e^{-2b}\left(\frac{\partial}{\partial z}\right)^2 E(z,t;0,b)-M^2E(z,t;0,b)\right].
\end{aligned}
$$

In the last integral we change the order of integration,

$$
\begin{aligned}
&u(x,t)\\
=\;&\int_0^{1-e^{-t}}\Big[\frac{1}{4}e^{t/2}\big(2+\ln(z+e^{-t})\big)+\frac{1}{16}(1+4M^2)e^{3t/2}z\ln(z+e^{-t})\Big]\frac{1}{\sqrt{z+e^{-t}}}\Big[\varphi(x+z)+\varphi(x-z)\Big]dz\\
&+\int_0^{1-e^{-t}} dz\Big[\varphi(x+z)+\varphi(x-z)\Big]\int_0^{-\ln(z+e^{-t})} db\,b\left[e^{-2b}\left(\frac{\partial}{\partial z}\right)^2 E(z,t;0,b)-M^2E(z,t;0,b)\right],
\end{aligned}
$$

and obtain (4.1), where $K_1(z,t)$ is defined by (4.5). Corollary is proven. $\square$

Proof of Theorem 0.4 with $\varphi_0=0$. The next lemma completes the proof of Theorem 0.4.

Lemma 4.3 *The kernel $K_1(z,t)$ defined by (4.5) coincides with one given in Theorem 0.4.*

Proof. We have due to Lemma 1.2, (3.3), and by integration by parts

$$
\begin{aligned}
&\int_0^{-\ln(z+e^{-t})} b\left[e^{-2b}\left(\frac{\partial}{\partial z}\right)^2 E(z,t;0,b)-M^2E(z,t;0,b)\right]db=\int_0^{-\ln(z+e^{-t})} b\Big(\frac{\partial}{\partial b}\Big)^2E(z,t;0,b)db\\
=\;&(-\ln(z+e^{-t}))\Big[\frac{\partial}{\partial b}E(z,t;0,b)\Big]_{b=-\ln(z+e^{-t})}-E(z,t;0,-\ln(z+e^{-t}))+E(z,t;0,0)\,.
\end{aligned}
$$

On the other hand, (3.4) and (3.9) of Proposition 3.3 imply

$$
\begin{aligned}
&\int_0^{-\ln(z+e^{-t})} db\,b\left[e^{-2b}\left(\frac{\partial}{\partial z}\right)^2 E(z,t;0,b)-M^2E(z,t;0,b)\right]\\
=\;&-\ln(z+e^{-t})\frac{1}{16}\frac{4+e^tz(1+4M^2)}{\sqrt{e^{-t}}\sqrt{z+e^{-t}}}-\frac{1}{2}\frac{1}{\sqrt{e^{-t}}\sqrt{z+e^{-t}}}+E(z,t;0,0).
\end{aligned}
$$

Thus, for the kernel $K_1(z,t)$ defined by (4.5) we have

$$\begin{aligned}
K_1(z,t) &= \Big[\frac{1}{4}e^{t/2}\big(2+\ln(z+e^{-t})\big)+\frac{1}{16}(1+4M^2)e^{3t/2}z\ln(z+e^{-t})\Big]\frac{1}{\sqrt{z+e^{-t}}}\\
&\quad -\ln(z+e^{-t})\frac{1}{16}\frac{4+e^t z(1+4M^2)}{\sqrt{e^{-t}}\sqrt{z+e^{-t}}}-\frac{1}{2}\frac{1}{\sqrt{e^{-t}}\sqrt{z+e^{-t}}}+E(z,t;0,0)\\
&= e^{t/2}\Big[\frac{1}{4}\big(2+\ln(z+e^{-t})\big)+\frac{1}{16}(1+4M^2)e^t z\ln(z+e^{-t})\\
&\quad -\ln(z+e^{-t})\frac{1}{16}(4+e^t z(1+4M^2))-\frac{1}{2}\Big]\frac{1}{\sqrt{z+e^{-t}}}+E(z,t;0,0)\\
&= E(z,t;0,0)\,.
\end{aligned}$$

The last line coincides with $K_1(z,t)$ of Theorem 0.4. Lemma is proven. □

5 The Cauchy Problem: First Datum and $n = 1$

In this section we prove Theorem 0.4 in the case of $\varphi_1(x)=0$. Thus, we have to prove the representation given by Theorem 0.4 for the solution $u=u(x,t)$ of the Cauchy problem (0.20) with $\varphi_1(x)=0$, that is equivalent to

$$u(x,t)=\frac{1}{2}e^{\frac{t}{2}}\Big[\varphi_0(x+1-e^{-t})+\varphi_0(x-1+e^{-t})\Big]+\int_0^1\big[\varphi_0(x-\phi(t)s)+\varphi_0(x+\phi(t)s)\big]K_0(\phi(t)s,t)\phi(t)\,ds,$$

where $\phi(t)=1-e^{-t}$. The proof of this case consists of the several steps.

Proposition 5.1 *The solution $u=u(x,t)$ of the Cauchy problem (0.20) can be represented as follows*

$$\begin{aligned}
&u(x,t) \qquad\qquad (5.1)\\
=\;& \frac{1}{2}e^{\frac{t}{2}}\Big[\varphi_0(x+1-e^{-t})+\varphi_0(x-1+e^{-t})\Big]\\
&-\int_0^t\Big[\frac{1}{4}e^{t/2}e^{-b/2}+\frac{1}{16}(1+4M^2)e^{t/2}e^{-3b/2}(e^t-e^b)\Big]\Big[\varphi_0(x+e^{-b}-e^{-t})+\varphi_0(x-e^{-b}+e^{-t}))\Big]\,db\\
&+\int_0^t db\int_{x-(e^{-b}-e^{-t})}^{x+e^{-b}-e^{-t}}dy\,\varphi_0(y)\left[e^{-2b}\left(\frac{\partial}{\partial y}\right)^2E(x-y,t;0,b)-M^2E(x-y,t;0,b)\right].
\end{aligned}$$

Proof. We set $u(x,t)=w(x,t)+\varphi_0(x)$, then

$$w_{tt}-e^{-2t}w_{xx}+M^2w=e^{-2t}\varphi_{0,xx}-M^2\varphi_0(x)\,,\qquad w(x,0)=0\,,\qquad w_t(x,0)=0\,.$$

Next we plug $f(x,t)=e^{-2t}\varphi_{0,xx}(x)-M^2\varphi_0(x)$ in the formula given by Theorem 0.3 and obtain

$$w(x,t) \;=\; \widetilde{w(x,t)}-\int_0^t db\int_{x-e^{-b}+e^{-t}}^{x+e^{-b}-e^{-t}}dy\,M^2\varphi_0(y)E(x-y,t;0,b), \tag{5.2}$$

where we have denoted

$$\widetilde{w(x,t)} \;:=\; \int_0^t db\int_{x-e^{-b}+e^{-t}}^{x+e^{-b}-e^{-t}}dy\,e^{-2b}\varphi_0^{(2)}(y)E(x-y,t;0,b)\,.$$

Next we integrate by parts and apply (3.3):

$$\begin{aligned}
\widetilde{w(x,t)} &= \int_0^t e^{-2b}\Big[\varphi_0^{(1)}(x+e^{-b}-e^{-t})-\varphi_0^{(1)}(x-e^{-b}+e^{-t})\Big]E(e^{-b}-e^{-t},t;0,b)\,db\\
&\quad-\int_0^t db\int_{x-e^{-b}+e^{-t}}^{x+e^{-b}-e^{-t}}dy\,e^{-2b}\varphi_0^{(1)}(y)\frac{\partial}{\partial y}E(x-y,t;0,b)\,.
\end{aligned}$$

On the other hand,

$$\varphi_0^{(1)}(x+e^{-b}-e^{-t}) = -e^b\frac{\partial}{\partial b}\varphi_0(x+e^{-b}-e^{-t}) \quad \text{and} \quad \varphi_0^{(1)}(x-e^{-b}+e^{-t}) = e^b\frac{\partial}{\partial b}\varphi_0(x-e^{-b}+e^{-t})$$

imply

$$\begin{aligned}\widetilde{w(x,t)} &= -\int_0^t e^{-b}\Big[\frac{\partial}{\partial b}\varphi_0(x+e^{-b}-e^{-t}) + \frac{\partial}{\partial b}\varphi_0(x-e^{-b}+e^{-t})\Big]E(e^{-b}-e^{-t},t;0,b)\,db \\ &\quad -\int_0^t db\int_{x-e^{-b}+e^{-t}}^{x+e^{-b}-e^{-t}} dy\, e^{-2b}\varphi_0^{(1)}(y)\frac{\partial}{\partial y}E(x-y,t;0,b)\,.\end{aligned}$$

One more integration by parts leads to

$$\begin{aligned}\widetilde{w(x,t)}+\varphi_0(x) &= \frac{1}{2}e^{\frac{t}{2}}\Big[\varphi_0(x+1-e^{-t})+\varphi_0(x-1+e^{-t})\Big] \\ &\quad +\int_0^t \Big[\varphi_0(x+e^{-b}-e^{-t})+\varphi_0(x-e^{-b}+e^{-t})\Big]\frac{\partial}{\partial b}\Big(e^{-b}E(e^{-b}-e^{-t},t;0,b)\Big)\,db \\ &\quad -\int_0^t db\int_{x-e^{-b}+e^{-t}}^{x+e^{-b}-e^{-t}} dy\, e^{-2b}\varphi_0^{(1)}(y)\frac{\partial}{\partial y}E(x-y,t;0,b)\,,\end{aligned}$$

where $E(0,t;0,t)=\frac{1}{2}e^t$ and $E(1-e^{-t},t;0,0)=\frac{1}{2}e^{\frac{t}{2}}$ have been used. Next we apply (3.5) of Proposition 3.3 and an integration by parts to obtain

$$\begin{aligned}\widetilde{w(x,t)}+\varphi_0(x) &= \frac{1}{2}e^{\frac{t}{2}}\Big[\varphi_0(x+1-e^{-t})+\varphi_0(x-1+e^{-t})\Big] \\ &\quad -\int_0^t \frac{1}{4}e^{t/2}e^{-b/2}\Big[\varphi_0(x+e^{-b}-e^{-t})+\varphi_0(x-e^{-b}+e^{-t})\Big]\,db \\ &\quad +\int_0^t db\, e^{-2b}\Big[\varphi_0(y)\frac{\partial}{\partial x}E(x-y,t;0,b)\Big]_{y=x-e^{-b}+e^{-t}}^{y=x+e^{-b}-e^{-t}} \\ &\quad +\int_0^t db\int_{x-e^{-b}+e^{-t}}^{x+e^{-b}-e^{-t}} dy\,\varphi_0(y)e^{-2b}\left(\frac{\partial}{\partial y}\right)^2 E(x-y,t;0,b)\,.\end{aligned}$$

We have due to (3.7) and (3.8) of Proposition 3.3

$$\begin{aligned}&\widetilde{w(x,t)}+\varphi_0(x) \\ =\; & \frac{1}{2}e^{\frac{t}{2}}\Big[\varphi_0(x+1-e^{-t})+\varphi_0(x-1+e^{-t})\Big] \\ & -\int_0^t \Big[\frac{1}{4}e^{t/2}e^{-b/2}+\frac{1}{16}(1+4M^2)e^{t/2}e^{-3b/2}(e^t-e^b)\Big]\Big[\varphi_0(x+e^{-b}-e^{-t})+\varphi_0(x-e^{-b}+e^{-t}))\Big]\,db \\ & +\int_0^t db\int_{x-e^{-b}+e^{-t}}^{x+e^{-b}-e^{-t}} dy\,\varphi_0(y)e^{-2b}\left(\frac{\partial}{\partial y}\right)^2 E(x-y,t;0,b)\,.\end{aligned}$$

Then the last equation together with (5.2) proves the desired representation. Proposition is proven. $\square$

Completion of the proof of Theorem 0.4. We make change $z=e^{-b}-e^{-t}$, $dz=-e^{-b}db$, and $b=-\ln(z+e^t)$ in the second term of the representation given by the previous proposition:

$$\begin{aligned}&\int_0^t \Big[\frac{1}{4}e^{t/2}e^{-b/2}+\frac{1}{16}(1+4M^2)e^{t/2}e^{-3b/2}(e^t-e^b)\Big]\Big[\varphi_0(x+e^{-b}-e^{-t})+\varphi_0(x-e^{-b}+e^{-t}))\Big]\,db \\ =\; & \int_0^{1-e^{-t}} \Big[\frac{1}{4}e^{t/2}\sqrt{z+e^{-t}}+\frac{1}{16}(1+4M^2)e^{t/2}(\sqrt{z+e^{-t}})^3\frac{ze^t}{z+e^{-t}}\Big]\Big[\varphi_0(x+z)+\varphi_0(x-z)\Big]\frac{1}{z+e^{-t}}\,dz \\ =\; & \int_0^{1-e^{-t}} e^t\Big[\frac{1}{4}+\frac{1}{16}(1+4M^2)ze^t\Big]\frac{1}{\sqrt{e^tz+1}}\Big[\varphi_0(x+z)+\varphi_0(x-z)\Big]\,dz\,.\end{aligned}$$

Next we apply (3.3) to the last term of that representation, and then we change the order of integration:

$$
\begin{aligned}
& \int_0^t db \int_{x-(e^{-b}-e^{-t})}^{x+e^{-b}-e^{-t}} dy\, \varphi_0(y) \left[e^{-2b} \left(\frac{\partial}{\partial y} \right)^2 E(x-y,t;0,b) - M^2 E(x-y,t;0,b) \right] \\
= & \int_0^t db \int_0^{e^{-b}-e^{-t}} dz \left[\varphi_0(x+z) + \varphi_0(x-z) \right] \left[e^{-2b} \left(\frac{\partial}{\partial z} \right)^2 E(z,t;0,b) - M^2 E(z,t;0,b) \right] \\
= & \int_0^{1-e^{-t}} dz \Big[\varphi_0(x+z) + \varphi_0(x-z) \Big] \int_0^{-\ln(z+e^{-t})} db \left[e^{-2b} \left(\frac{\partial}{\partial z} \right)^2 E(z,t;0,b) - M^2 E(z,t;0,b) \right] .
\end{aligned}
$$

On the other hand, since the function $E(z,t;0,b)$ solves Klein-Gordon equation, the last integral is equal to

$$
\begin{aligned}
& \int_0^{1-e^{-t}} dz \Big[\varphi_0(x+z) + \varphi_0(x-z) \Big] \int_0^{-\ln(z+e^{-t})} db \left[\left(\frac{\partial}{\partial t} \right)^2 E(z,t;0,b) \right] \\
= & \int_0^{1-e^{-t}} dz \Big[\varphi_0(x+z) + \varphi_0(x-z) \Big] \left[\frac{\partial}{\partial b} E(z,t;0,b) \right]_{b=0}^{b=-\ln(z+e^{-t})} .
\end{aligned}
$$

Application of (3.9) and (3.10) completes the proof. Theorem 0.4 is proven. $\square$

6 n-Dimensional Case, $n \geq 2$

Proof of Theorem 0.5. Let us consider the case of $x \in \mathbb{R}^n$, where $n = 2m+1,\ m \in \mathbb{N}$. First for the given function $u = u(x,t)$ we define the spherical means of u about point x:

$$
I_u(x,r,t) \quad = \quad \frac{1}{\omega_{n-1}} \int_{S^{n-1}} u(x+ry,t)\, dS_y \, ,
$$

where ω_{n-1} denotes the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Then we define an operator Ω_r by

$$
\Omega_r(u)(x,t) := \Big(\frac{1}{r} \frac{\partial}{\partial r} \Big)^{m-1} r^{2m-1} I_u(x,r,t) \, .
$$

One can show that there are constants $c_j^{(n)}$, $j = 0, \ldots, m-1$, where $n = 2m+1$, with $c_0^{(n)} = 1 \cdot 3 \cdot 5 \cdots (n-2)$, such that

$$
\Big(\frac{1}{r} \frac{\partial}{\partial r} \Big)^{m-1} r^{2m-1} \varphi(r) = r \sum_{j=0}^{m-1} c_j^{(n)} r^j \frac{\partial^j}{\partial r^j} \varphi(r) \, .
$$

One can recover the functions according to

$$
u(x,t) \quad = \quad \lim_{r \to 0} I_u(x,r,t) = \lim_{r \to 0} \frac{1}{c_0^{(n)} r} \Omega_r(u)(x,t) \, , \tag{6.1}
$$

$$
u(x,0) \quad = \quad \lim_{r \to 0} \frac{1}{c_0^{(n)} r} \Omega_r(u)(x,0) \, , \quad u_t(x,0) = \lim_{r \to 0} \frac{1}{c_0^{(n)} r} \Omega_r(\partial_t u)(x,0) \, . \tag{6.2}
$$

It is well known that $\Delta_x \Omega_r h = \frac{\partial^2}{\partial r^2} \Omega_r h$ for every function $h \in C^2(\mathbb{R}^n)$. Therefore we arrive at the following mixed problem for the function $v(x,r,t) := \Omega_r(u)(x,r,t)$:

$$
\begin{cases}
v_{tt}(x,r,t) - e^{-2t} v_{rr}(x,r,t) + M^2 v(x,r,t) = F(x,r,t) \quad \text{for all} \quad t \geq 0 \, , \ r \geq 0 \, , \ x \in \mathbb{R}^n \, , \\
v(x,0,t) = 0 \quad \text{for all} \quad t \geq 0 \, , \quad x \in \mathbb{R}^n, \\
v(x,r,0) = 0 \, , \quad v_t(x,r,0) = 0 \quad \text{for all} \quad r \geq 0 \, , \quad x \in \mathbb{R}^n \, , \\
F(x,r,t) := \Omega_r(f)(x,t) \, , \quad F(x,0,t) = 0 \, , \quad \text{for all} \quad x \in \mathbb{R}^n \, .
\end{cases}
$$

It must be noted here that the spherical mean I_u defined for $r > 0$ has an extension as even function for $r < 0$ and hence $\Omega_r(u)$ has a natural extension as an odd function. That allows replacing the mixed problem

with the Cauchy problem. Namely, let functions $\widetilde{v}$ and $\widetilde{F}$ be the continuations of the functions v and F, respectively, by

$$\widetilde{v}(x,r,t)=\begin{cases} v(x,r,t), \ if \ r\geq 0\\ -v(x,-r,t), \ if \ r\leq 0\end{cases}, \quad \widetilde{F}(x,r,t)=\begin{cases} F(x,r,t), \ if \ r\geq 0\\ -F(x,-r,t), \ if \ r\leq 0\end{cases}.$$

Then $\widetilde{v}$ solves the Cauchy problem

$$\begin{cases} \widetilde{v}_{tt}(x,r,t)-e^{-2t}\widetilde{v}_{rr}(x,r,t)+M^2\widetilde{v}(x,r,t)=\widetilde{F}(x,r,t) \quad \text{for all} \quad t\geq 0\,, \quad r\in\mathbb{R}\,, \quad x\in\mathbb{R}^n\,,\\ \widetilde{v}(x,r,0)=0\,, \quad \widetilde{v}_t(x,r,0)=0 \quad \text{for all} \quad r\in\mathbb{R}\,, \quad x\in\mathbb{R}^n. \end{cases}$$

Hence, according to Theorem 0.3, one has the representation

$$\begin{aligned}\widetilde{v}(x,r,t) &= \int_0^t db\int_{r-(e^{-b}-e^{-t})}^{r+e^{-b}-e^{-t}} dr_1\, \widetilde{F}(x,r_1,b)(4e^{-b-t})^{iM}\left((e^{-t}+e^{-b})^2-(r-r_1)^2\right)^{-\frac{1}{2}-iM}\\ &\qquad\times F\left(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-b}-e^{-t})^2-(r-r_1)^2}{(e^{-b}+e^{-t})^2-(r-r_1)^2}\right).\end{aligned}$$

Since $u(x,t)=\lim_{r\to 0}\left(\widetilde{v}(x,r,t)/(c_0^{(n)}r)\right)$, we consider the case of $r<t$ in the above representation to obtain:

$$\begin{aligned}u(x,t) &= \frac{1}{c_0^{(n)}}\int_0^t db\int_0^{e^{-b}-e^{-t}} dr_1\,\lim_{r\to 0}\frac{1}{r}\left\{\widetilde{F}(x,r+r_1,b)+\widetilde{F}(x,r-r_1,b)\right\}\\ &\qquad\times(4e^{-b-t})^{iM}\left((e^{-t}+e^{-b})^2-r_1^2\right)^{-\frac{1}{2}-iM} F\left(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-b}-e^{-t})^2-r_1^2}{(e^{-b}+e^{-t})^2-r_1^2}\right).\end{aligned}$$

Then by definition of the function $\widetilde{F}$, we replace $\lim_{r\to 0}\frac{1}{r}\left\{\widetilde{F}(x,r-r_1,b)+\widetilde{F}(x,r+r_1,b)\right\}$ with $2\left(\frac{\partial}{\partial r}F(x,r,b)\right)_{r=r_1}$ in the last formula. The definitions of $F(x,r,t)$ and of the operator Ω_r yield:

$$\begin{aligned}u(x,t) &= \frac{2}{c_0^{(n)}}\int_0^t db\int_0^{e^{-b}-e^{-t}} dr_1\,\left(\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{m-1}r^{2m-1}I_f(x,r,t)\right)_{r=r_1}\\ &\qquad\times(4e^{-b-t})^{iM}\left((e^{-t}+e^{-b})^2-r_1^2\right)^{-\frac{1}{2}-iM} F\left(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{-b}-e^{-t})^2-r_1^2}{(e^{-b}+e^{-t})^2-r_1^2}\right),\end{aligned}$$

where $x\in\mathbb{R}^n$, $n=2m+1$, $m\in\mathbb{N}$. Thus, the solution to the Cauchy problem is given by (0.22). We employ the method of descent to complete the proof for the case of even n, $n=2m$, $m\in\mathbb{N}$. Theorem 0.5 is proven. $\square$

Proof of (0.16) and (0.17). We set $f(x,b)=\delta(x)\delta(t-t_0)$ in (0.22) and (0.23), and we obtain (0.16) and (0.17), where if n is odd, then

$$E^w(x,t):=\frac{1}{\omega_{n-1}1\cdot 3\cdot 5\ldots\cdot(n-2)}\frac{\partial}{\partial t}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\frac{1}{t}\delta(|x|-t)\,,$$

while for n even we have

$$E^w(x,t):=\frac{2}{\omega_{n-1}1\cdot 3\cdot 5\ldots\cdot(n-1)}\frac{\partial}{\partial t}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}\frac{1}{\sqrt{t^2-|x|^2}}\chi_{B_t(x)}\,.$$

Here $\chi_{B_t(x)}$ denotes the characteristic function of the ball $B_t(x):=\{x\in\mathbb{R}^n;\ |x|\leq t\}$. Constant ω_{n-1} is the area of the unit sphere $S^{n-1}\subset\mathbb{R}^n$. The distribution $\delta(|x|-t)$ is defined by $<\delta(|\cdot|-t),f(\cdot)>=\int_{|x|=t}f(x)\,dx$ for $f\in C_0^\infty(\mathbb{R}^n)$.

Proof of Theorem 0.6. First we consider case of $\varphi_0(x)=0$. More precisely, we have to prove that the solution $u(x,t)$ of the Cauchy problem (0.25) with $\varphi_0(x)=0$ can be represented by (0.26) with $\varphi_0(x)=0$. The next lemma will be used in both cases.

Lemma 6.1 *Consider the mixed problem*

$$\begin{cases} v_{tt} - e^{-2t}v_{rr} + M^2 v = 0 \quad \text{for all} \quad t \geq 0\,, \quad r \geq 0\,, \\ v(r,0) = \tau_0(r)\,, \quad v_t(r,0) = \tau_1(r) \quad \text{for all} \quad r \geq 0\,, \\ v(0,t) = 0 \quad \text{for all} \quad t \geq 0\,, \end{cases}$$

and denote by $\widetilde{\tau}_0(r)$ *and* $\widetilde{\tau}_1(r)$ *the continuations of the functions* $\tau_0(r)$ *and* $\tau_1(r)$ *for negative* r *as odd functions:* $\widetilde{\tau}_0(-r) = -\tau_0(r)$ *and* $\widetilde{\tau}_1(-r) = -\tau_1(r)$ *for all* $r \geq 0$*, respectively. Then solution* $v(r,t)$ *to the mixed problem is given by the restriction of (4.1) to* $r \geq 0$*:*

$$\begin{aligned} v(r,t) \;&=\; \frac{1}{2}e^{\frac{t}{2}}\Big[\widetilde{\tau}_0(r+1-e^{-t}) + \widetilde{\tau}_0(r-1+e^{-t})\Big] + \int_0^1 \big[\widetilde{\tau}_0(r-\phi(t)s) + \widetilde{\tau}_0(r+\phi(t)s)\big] K_0(\phi(t)s,t)\phi(t)\, ds \\ &\quad + \int_0^1 \Big[\widetilde{\tau}_1\Big(r+\phi(t)s\Big) + \widetilde{\tau}_1\Big(r-\phi(t)s\Big)\Big] K_1(\phi(t)s,t)\phi(t)\, ds\,, \end{aligned}$$

where $K_0(z,t)$ *and* $K_1(z,t)$ *are defined in Theorem 0.4 and* $\phi(t) = 1 - e^{-t}$*.*

Proof. This lemma is a direct consequence of Theorem 0.4. □

Now let us consider the case of $x \in \mathbb{R}^n$, where $n = 2m+1$. First for the given function $u = u(x,t)$ we define the spherical means of u about point x. One can recover the functions by means of (6.1), (6.2), and

$$\varphi_i(x) \;=\; \lim_{r\to 0} I_{\varphi_i}(x,r) = \lim_{r\to 0} \frac{1}{c_0^{(n)} r}\Omega_r(\varphi_i)(x)\,, \quad i = 0,1\,.$$

Then we arrive at the following mixed problem

$$\begin{cases} v_{tt}(x,r,t) - e^{-2t}v_{rr}(x,r,t) + M^2 v(x,r,t) = 0 \quad \text{for all} \quad t \geq 0\,,\; r \geq 0\,,\; x \in \mathbb{R}^n\,, \\ v(x,0,t) = 0 \quad \text{for all} \quad t \geq 0\,, \quad x \in \mathbb{R}^n\,, \\ v(x,r,0) = 0\,, \quad v_t(x,r,0) = \Phi_1(x,r) \quad \text{for all} \quad r \geq 0\,, \quad x \in \mathbb{R}^n\,, \end{cases}$$

with the unknown function $v(x,r,t) := \Omega_r(u)(x,r,t)$, where

$$\Phi_i(x,r) := \Omega_r(\varphi_i)(x) = \Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{m-1} r^{2m-1}\frac{1}{\omega_{n-1}}\int_{S^{n-1}} \varphi_i(x+ry)\, dS_y\,, \tag{6.3}$$

$$\Phi_i(x,0) = 0\,, \quad i = 0,1, \qquad \text{for all} \quad x \in \mathbb{R}^n\,. \tag{6.4}$$

Then, according to Lemma 6.1 and to $u(x,t) = \lim_{r\to 0}\big(v(x,r,t)/(c_0^{(n)}r)\big)$, we obtain:

$$u(x,t) \;=\; \frac{1}{c_0^{(n)}}\lim_{r\to 0}\frac{1}{r}\int_0^1 \Big[\widetilde{\Phi}_1\big(x,r+\phi(t)s\big) + \widetilde{\Phi}_1\big(x,r-\phi(t)s\big)\Big] K_1(\phi(t)s,t)\phi(t)\, ds\,.$$

The last limit is equal to

$$\begin{aligned} & 2\int_0^1 \Big(\frac{\partial}{\partial r}\Phi_1(x,r)\Big)_{r=\phi(t)s} K_1(\phi(t)s,t)\phi(t)\, ds \\ =\;\; & 2\int_0^1 \bigg(\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{\frac{n-3}{2}} \frac{r^{n-2}}{\omega_{n-1}}\int_{S^{n-1}} \varphi_1(x+ry)\, dS_y\bigg)_{r=\phi(t)s} K_1(\phi(t)s,t)\phi(t)\, ds\,. \end{aligned}$$

Thus, Theorem 0.6 in the case of $\varphi_0(x) = 0$ is proven.

Now we turn to the case of $\varphi_1(x) = 0$. Thus, we arrive at the following mixed problem

$$\begin{cases} v_{tt}(x,r,t) - e^{-2t}v_{rr}(x,r,t) + M^2 v(x,r,t) = 0 \quad \text{for all} \quad t \geq 0\,,\; r \geq 0\,,\; x \in \mathbb{R}^n\,, \\ v(x,r,0) = \Phi_0(x,r)\,, \quad v_t(x,r,0) = 0 \quad \text{for all} \quad r \geq 0\,, \quad x \in \mathbb{R}^n\,, \\ v(x,0,t) = 0 \quad \text{for all} \quad t \geq 0\,, \quad x \in \mathbb{R}^n\,, \end{cases}$$

with the unknown function $v(x,r,t) := \Omega_r(u)(x,r,t)$ defined by (6.3), (6.4). Then, according to Lemma 6.1 and to $u(x,t) = \lim_{r\to 0}\big(v(x,r,t)/(c_0^{(n)} r)\big)$, we obtain:

$$
\begin{aligned}
u(x,t) &= \frac{1}{c_0^{(n)}} e^{\frac{t}{2}} \lim_{r\to 0} \frac{1}{2r}\Big[\widetilde{\Phi}_0(x, r+e^t-1) + \widetilde{\Phi}_0(x, r-e^t+1)\Big] \\
&\quad + \frac{2}{c_0^{(n)}} \int_0^1 \lim_{r\to 0} \frac{1}{2r}\big[\widetilde{\Phi}_0(x, r-\phi(t)s) + \widetilde{\Phi}_0(x, r+\phi(t)s)\big] K_0(\phi(t)s,t)\phi(t)\, ds\,, \\
&= \frac{1}{c_0^{(n)}} e^{\frac{t}{2}} \left(\frac{\partial}{\partial r}\Phi_0(x,r)\right)_{r=\phi(t)} + \frac{2}{c_0^{(n)}} \int_0^1 \left(\frac{\partial}{\partial r}\Phi_0(x,r)\right)_{r=\phi(t)s} K_0(\phi(t)s,t)\phi(t)\, ds \\
&= e^{\frac{t}{2}} v_{\varphi_0}(x,\phi(t)) + 2\int_0^1 v_{\varphi_0}(x,\phi(t)s) K_0(\phi(t)s,t)\phi(t)\, ds\,.
\end{aligned}
$$

Theorem 0.6 is proven. □

7 L^p–L^q and L^q–L^q Estimates for the Solutions of One-dimensional Equation, $n=1$

Consider now the Cauchy problem for the equation (0.18) with the source term and with vanishing initial data (0.19). In this and next sections we restrict ourselves to the particles with "large" mass $m \ge n/2$, that is, with nonnegative curved mass $M \ge 0$, to make presentation more transparent. The case of $m < n/2$ will be discussed in the forthcoming paper.

Theorem 7.1 *For every function $f \in C^2(\mathbb{R}\times[0,\infty))$ such that $f(\cdot,t) \in C_0^\infty(\mathbb{R}_x)$ the solution $u = u(x,t)$ of the Cauchy problem (0.18), (0.19) satisfies inequality*

$$
\|u(x,t)\|_{L^q(\mathbb{R}_x)} \le C_M e^{t(1-1/\rho)} \int_0^t (1+t-b)(e^{t-b}-1)^{1/\rho}(e^{t-b}+1)^{-1} \|f(x,b)\|_{L^p(\mathbb{R}_x)}\, db
$$

for all $t>0$, where $1<p<\rho'$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{\rho'}$, $\rho < 2$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$.

Proof. Using the fundamental solution from Theorem 0.1 one can write the convolution

$$
u(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{E}_+(x,t;y,b) f(y,b)\, db\, dy = \int_0^t db \int_{-\infty}^{\infty} \mathcal{E}_+(x-y,t;0,b) f(y,b)\, dy\,.
$$

Due to Young's inequality we have

$$
\|u(x,t)\|_{L^q(\mathbb{R}_x)} \le c_k \int_0^t db \left(\int_{-(\phi(t)-\phi(b))}^{\phi(t)-\phi(b)} |E(x,t;0,b)|^\rho dx \right)^{1/\rho} \|f(x,b)\|_{L^p(\mathbb{R}_x)},
$$

where $1<p<\rho'$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{\rho'}$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$, $\phi(t) = 1-e^{-t}$. The integral in parentheses can be transformed as follows

$$
\begin{aligned}
&\int_{-(\phi(t)-\phi(b))}^{\phi(t)-\phi(b)} |E(x,t;0,b)|^\rho dx \\
&= 2e^{-t+t\rho} \int_0^{e^{t-b}-1} \Big((e^{t-b}+1)^2 - y^2\Big)^{-\frac{\rho}{2}} \left| F\Big(\frac{1}{2}+iM, \frac{1}{2}+iM; 1; \frac{(e^{t-b}-1)^2-y^2}{(e^{t-b}+1)^2-y^2}\Big) \right|^\rho dy.
\end{aligned}
$$

On the other hand, due to integral representation for the hypergeometric function (1)[2, v.1, Sec.2.12] for $\zeta \in [0,1)$ one has

$$
\left| F\Big(\frac{1}{2}+iM, \frac{1}{2}+iM; 1; \zeta\Big) \right| \le \frac{1}{|\Gamma\left(\frac{1}{2}+iM\right)|^2} \pi F\Big(\frac{1}{2}, \frac{1}{2}; 1; \zeta\Big)\,. \tag{7.1}
$$

Thus,

$$\int_{-(\phi(t)-\phi(b))}^{\phi(t)-\phi(b)} |E(x,t;0,b)|^\rho dx \le C_M e^{-t+t\rho} \int_0^{e^{t-b}-1} \left((e^{t-b}+1)^2-y^2\right)^{-\frac{\rho}{2}} \left|F\Big(\frac{1}{2},\frac{1}{2};1;\frac{(e^{t-b}-1)^2-y^2}{(e^{t-b}+1)^2-y^2}\Big)\right|^\rho dy.$$

Lemma 7.2 [33] *For all $z>1$ the following estimate*

$$\int_0^{z-1} ((z+1)^2-r^2)^{-\frac{\rho}{2}} F\left(\frac{1}{2},\frac{1}{2};1;\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\right)^\rho dr \le C(1+\ln z)^\rho (z-1)(z+1)^{-\rho} F\Big(\frac{1}{2},\frac{\rho}{2};\frac{3}{2};\frac{(z-1)^2}{(z+1)^2}\Big)$$

is fulfilled, provided that $1<p<\rho'$, $\frac{1}{q}=\frac{1}{p}-\frac{1}{\rho'}$, $\frac{1}{\rho}+\frac{1}{\rho'}=1$. In particular, if $\rho<2$, then

$$\int_0^{z-1} ((z+1)^2-r^2)^{-\frac{\rho}{2}} F\left(\frac{1}{2},\frac{1}{2};1;\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\right)^\rho dr \quad \le \quad C(1+\ln z)^\rho (z-1)(z+1)^{-\rho}\,.$$

Completion of the proof of Theorem 7.1. Thus for $\rho<2$ and $z=e^{t-b}$ we have

$$\|u(x,t)\|_{L^q(\mathbb{R}_x)} \quad \le \quad C_M e^{t(1-1/\rho)} \int_0^t (1+t-b)(e^{t-b}-1)^{1/\rho}(e^{t-b}+1)^{-1}\|f(x,b)\|_{L^p(\mathbb{R}_x)}\, db\,.$$

The last inequality implies the estimate of the statement of theorem. Theorem 7.1 is proven. □

Proposition 7.3 *The solution $u=u(x,t)$ of the Cauchy problem*

$$u_{tt}-e^{-2t}u_{xx}+M^2u=0\,,\qquad u(x,0)=\varphi_0(x)\,,\qquad u_t(x,0)=\varphi_1(x)\,,$$

with φ_0, $\varphi_1\in C_0^\infty(\mathbb{R})$ satisfies the following estimate

$$\|u(x,t)\|_{L^q(\mathbb{R}_x)} \le C(1+t)\Big(e^{\frac{t}{2}}\|\varphi_0(x)\|_{L^q(\mathbb{R}_x)}+(e^t-1)e^{-t}\|\varphi_1(x)\|_{L^q(\mathbb{R}_x)}\Big) \quad \textit{for all} \quad t\in(0,\infty). \tag{7.2}$$

Proof. First we consider the equation without source term but with the second datum that is the case of $\varphi_0=0$. For the convenience we drop subindex of φ_1. Then we apply the representation given by Theorem 0.4 for the solution $u=u(x,t)$ of the Cauchy problem with $\varphi_0=0$, and obtain

$$u(x,t) \quad = \quad \int_0^{1-e^{-t}} \Big[\varphi_1(x-z)+\varphi_1(x+z)\Big]K_1(z,t)dz\,,$$

where the kernel $K_1(z,t)$ is defined in Theorem 0.4. Hence, we arrive at inequality

$$\|u(x,t)\|_{L^q(\mathbb{R})} \le C_M\|\varphi(x)\|_{L^q(\mathbb{R})} \int_0^{1-e^{-t}} |K_1(r,t)|dr\,.$$

To estimate the last integral we denote it by I_1,

$$I_1(z) \quad := \quad \int_0^{1-e^{-t}} |K_1(r,t)|dr\,,$$

and with $z=e^t>1$ due to (7.1) we write

$$I_1(z) \quad \le \quad C\int_0^{z-1} \frac{1}{\sqrt{(1+z)^2-y^2}} F\left(\frac{1}{2},\frac{1}{2};1;\frac{y^2-(1-z)^2}{y^2-(1+z)^2}\right) dy\,. \tag{7.3}$$

Then, according to Lemma 7.2 (the case of $\rho=1$) we have for that integral the following estimate

$$I_1(e^t) \le C(1+t)(e^t-1)(e^t+1)^{-1}\,. \tag{7.4}$$

Finally, (7.3) and (7.4) imply the $L^q - L^q$ estimate (7.2) for the case of $\varphi_0 = 0$.

Next we consider the equation without source but with the first datum, that is, the case of $\varphi_1 = 0$. We apply the representation given by Theorem 0.4 for the solution $u = u(x,t)$ of the Cauchy problem with $\varphi_1 = 0$, and obtain

$$u(x,t) \;=\; \frac{1}{2}e^{\frac{t}{2}}\Big[\varphi_0(x+1-e^{-t}) + \varphi_0(x-1+e^{-t})\Big] + \int_0^{1-e^{-t}} \big[\varphi_0(x-r)+\varphi_0(x+r)\big]K_0(r,t)\,dr\,,$$

where the kernel $K_0(r,t)$ is defined in Theorem 0.4. Then we easily obtain the following two estimates:

$$\|u(x,t) - \int_0^{1-e^{-t}} \big[\varphi_0(x-r)+\varphi_0(x+r)\big]K_0(r,t)\,dr\|_{L^q(\mathbb{R})} \;\le\; e^{\frac{t}{2}}\|\varphi_0(x)\|_{L^q(\mathbb{R})}$$

and

$$\|u(x,t)\|_{L^q(\mathbb{R})} \;\le\; e^{\frac{t}{2}}\|\varphi_0(x)\|_{L^q(\mathbb{R})} + 2\|\varphi_0(x)\|_{L^q(\mathbb{R})}\int_0^{1-e^{-t}} |K_0(r,t)|\,dr\,.$$

Finally, the following lemma completes the proof of proposition.

Lemma 7.4 *The kernel $K_0(r,t)$ has an integrable singularity at $r = e^t - 1$, more precisely, one has*

$$\int_0^{1-e^{-t}} |K_0(r,t)|\,dr \le C(e^t-1)e^{-\frac{1}{2}t}(1+t) \quad \textit{for all} \quad t\in[0,\infty)\,.$$

Proof. For the integral we obtain

$$\begin{aligned}\int_0^{1-e^{-t}} |K_0(r,t)|\,dr \;\le\; & \int_0^{z-1} \frac{1}{[(z-1)^2-y^2]\sqrt{[(z+1)^2-y^2]}} \\ & \times\Big|\big(z-z^2-iM(1-z^2-y^2)\big)F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(z-1)^2-y^2}{(z+1)^2-y^2}\Big) \\ & +\big(z^2-1+y^2\big)\Big(\frac{1}{2}-iM\Big)F\Big(-\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(z-1)^2-y^2}{(z+1)^2-y^2}\Big)\Big|\,dy\end{aligned}$$

for all $z := e^t > 1$. We divide the domain of integration into two zones,

$$Z_1(\varepsilon,z) \;:=\; \Big\{(z,r)\,\Big|\,\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\le\varepsilon,\ 0\le r\le z-1\Big\}, \tag{7.5}$$

$$Z_2(\varepsilon,z) \;:=\; \Big\{(z,r)\,\Big|\,\varepsilon\le\frac{(z-1)^2-r^2}{(z+1)^2-r^2},\ 0\le r\le z-1\Big\}, \tag{7.6}$$

and split the integral into conformable two parts,

$$\int_0^{e^t-1} |K_0(r,t)|\,dr \;=\; \int_{(z,r)\in Z_1(\varepsilon,z)} |K_0(r,t)|\,dr + \int_{(z,r)\in Z_2(\varepsilon,z)} |K_0(r,t)|\,dr\,.$$

In the first zone we have

$$F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(z-1)^2-y^2}{(z+1)^2-y^2}\Big) = 1+\Big(\frac{1}{2}+iM\Big)^2\frac{(z-1)^2-y^2}{(z+1)^2-y^2}+O\left(\Big(\frac{(z-1)^2-y^2}{(z+1)^2-y^2}\Big)^2\right), \tag{7.7}$$

$$F\Big(-\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(z-1)^2-y^2}{(z+1)^2-y^2}\Big) = 1-\Big(\frac{1}{4}+M^2\Big)\frac{(z-1)^2-y^2}{(z+1)^2-y^2}+O\left(\Big(\frac{(z-1)^2-y^2}{(z+1)^2-y^2}\Big)^2\right). \tag{7.8}$$

We use the last formulas to estimate the term containing the hypergeometric functions:

$$\begin{aligned}
&\Big|\big(z-z^2-iM(1-z^2-r^2)\big)F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\Big)\\
&\qquad +\big(z^2-1+r^2\big)\Big(\frac{1}{2}-iM\Big)F\Big(-\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\Big)\Big|\\
\le\ & \frac{1}{2}\big[(z-1)^2-r^2\big]\\
&+\left|\big(z-z^2-iM(1-z^2-r^2)\big)\left(\frac{1}{2}+iM\right)^2-\big(z^2-1+r^2\big)\Big(\frac{1}{2}-iM\Big)\left(\frac{1}{4}+M^2\right)\right|\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\\
&+\Big(\big|z-z^2-iM(1-z^2-r^2)\big|+\big|z^2-1+y^2\big|\Big)O\left(\left(\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\right)^2\right)\\
=\ & \frac{1}{2}\big[(z-1)^2-r^2\big]\\
&+\frac{1}{8}\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\big|(1-2iM)(-1+4M^2)(y^2+z^2-1)+2(1+2iM)^2(-z^2+z+iM(y^2+z^2-1))\big|\\
&+\Big(\big|z-z^2-iM(1-z^2-r^2)\big|+\big|z^2-1+y^2\big|\Big)O\left(\left(\frac{(z-1)^2-y^2}{(z+1)^2-y^2}\right)^2\right). \qquad (7.9)
\end{aligned}$$

Hence, we have to consider the following three integrals, which can be easily evaluated and estimated,

$$\begin{aligned}
A_1 &:= \int_{(z,r)\in Z_1(\varepsilon,z)}\frac{1}{\sqrt{(z+1)^2-r^2}}dr \le \mathrm{Arctan}\left(\frac{z-1}{2\sqrt{z}}\right)\le\frac{\pi}{2}\,,\\
A_2 &:= \int_{(z,r)\in Z_1(\varepsilon,z)}\frac{z^2}{((z+1)^2-r^2)\sqrt{(z+1)^2-r^2}}dr \le (z+1)^{-1/2}(z-1),
\end{aligned}$$

and

$$A_3 := \int_{(z,r)\in Z_1(\varepsilon,z)}\frac{\big|z-z^2-iM(1-z^2-r^2)\big|+\big|z^2-1+r^2\big|}{\sqrt{(z+1)^2-r^2}}\frac{(z-1)^2-r^2}{((z+1)^2-r^2)^2}dr \le C_M(z+1)^{-1/2}(z-1)$$

for all $z\in[1,\infty)$. Finally, for the integral over the first zone we have obtained

$$\begin{aligned}
&\int_{(z,r)\in Z_1(\varepsilon,z)}dr\frac{1}{[(z-1)^2-r^2]\sqrt{[(z+1)^2-r^2]}}\\
&\times\Big|\big(z-z^2-iM(1-z^2-r^2)\big)F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\Big)\\
&\qquad +\big(z^2-1+r^2\big)\Big(\frac{1}{2}-iM\Big)F\Big(-\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\Big)\Big| \le C_M(z+1)^{-1/2}(z-1)
\end{aligned}$$

for all $z\in[1,\infty)$. In the second zone we have

$$\varepsilon\le\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\le 1 \quad\text{and}\quad \frac{1}{(z-1)^2-r^2}\le\frac{1}{\varepsilon[(z+1)^2-r^2]}\,. \qquad (7.10)$$

According to the formula 15.3.10 of [2, Ch.15] the hypergeometric functions obey the estimate

$$\Big|F\Big(-\frac{1}{2}+iM,\frac{1}{2}+iM;1;x\Big)\Big|\le C \text{ and } \Big|F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;x\Big)\Big|\le C\big(1-\ln(1-x)\big)\ \ \forall x\in[\varepsilon,1). \qquad (7.11)$$

This allows to estimate the integral over the second zone:

$$\int_{(z,r)\in Z_2(\varepsilon,z)} dr \frac{1}{[(z-1)^2-r^2]\sqrt{(z+1)^2-r^2}} \tag{7.12}$$
$$\times\Big|\big(z-z^2-iM(1-z^2-r^2)\big)F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\Big)$$
$$+\big(z^2-1+r^2\big)\Big(\frac{1}{2}-iM\Big)F\Big(-\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\Big)\Big| \le C_M(z+1)^{-1/2}(z-1)$$

for all $z\in[1,\infty)$. Indeed, for the argument of the hypergeometric functions we have

$$\varepsilon \le \frac{(z-1)^2-r^2}{(z+1)^2-r^2} = 1-\frac{4z}{(z+1)^2-r^2} < 1, \quad \frac{4z}{(z+1)^2-r^2} < 1-\varepsilon \quad \text{for all} \quad (z,r)\in Z_2(\varepsilon,z)\,. \tag{7.13}$$

Hence,

$$\Big|F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\Big)\Big| \le C\,(1+\ln z)\,,\ (z,r)\in Z_2(\varepsilon,z). \tag{7.14}$$

To prove (7.12) we estimate the following integral

$$\int_{(z,r)\in Z_2(\varepsilon,z)} \frac{z^2}{((z-1)^2-r^2)\sqrt{(z+1)^2-r^2}}dr \quad\le\quad C_\varepsilon z^2\int_0^{z-1}\frac{1}{((z+1)^2-r^2)^{3/2}}dr \;\le\; C_\varepsilon\frac{(z-1)}{\sqrt{z}}\,.$$

Thus, the lemma is proven. □

8 L^p-L^q Estimates for Equation with $n=1$ and without Source Term. Some Estimates of Kernels K_0 and K_1

Theorem 8.1 *Let $u=u(x,t)$ be a solution of the Cauchy problem*

$$u_{tt}-e^{-2t}u_{xx}+M^2u=0\,,\qquad u(x,0)=\varphi_0(x)\,,\qquad u_t(x,0)=\varphi_1(x)\,,$$

with φ_0, $\varphi_1\in C_0^\infty(\mathbb{R})$. If $\rho\in(1,2)$, then

$$\begin{aligned}\|u(x,t)\|_{L^q(\mathbb{R}_x)} \quad&\le\quad e^{\frac{t}{2}}\|\varphi_0(x)\|_{L^q(\mathbb{R}_x)}+C_\rho(1+t)(e^t-1)^{\frac{1}{\rho}}e^{t[\frac{1}{2}-\frac{1}{\rho}]}\|\varphi_0(x)\|_{L^p(\mathbb{R}_x)}\\ &\quad +C_\rho(1+t)(e^t-1)^{\frac{1}{\rho}}e^{-\frac{t}{\rho}}\|\varphi_1(x)\|_{L^p(\mathbb{R}_x)}\,,\end{aligned}$$

for all $t\in(0,\infty)$. Here $1<p<\rho'$, $\frac{1}{q}=\frac{1}{p}-\frac{1}{\rho'}$, $\frac{1}{\rho}+\frac{1}{\rho'}=1$. If $\rho=1$, then

$$\|u(x,t)\|_{L^q(\mathbb{R}_x)} \le C(1+t)\Big(e^{\frac{t}{2}}\|\varphi_0(x)\|_{L^q(\mathbb{R}_x)}+(e^t-1)e^{-t}\|\varphi_1(x)\|_{L^q(\mathbb{R}_x)}\Big)\,, \tag{8.1}$$

for all $t\in(0,\infty)$.

Proof. For $\rho=1$ we just apply Proposition 7.3. To prove this theorem for $\rho>1$ we need some auxiliary estimates for the kernels K_0 and K_1. We start with the case of $\varphi_0=0$, where the kernel K_1 appears. The application of Theorem 0.4 and Young's inequality lead to

$$\|u(x,t)\|_{L^q(\mathbb{R}_x)} \quad\le\quad 2\bigg(\int_0^{1-e^{-t}}|K_1(x,t)|^\rho dx\bigg)^{1/\rho}\|\varphi_1(x)\|_{L^p(\mathbb{R}_x)},$$

where $1<p<\rho'$, $\frac{1}{q}=\frac{1}{p}-\frac{1}{\rho'}$, $\frac{1}{\rho}+\frac{1}{\rho'}=1$. Now we have to estimate the last integral.

Proposition 8.2 *We have*

$$\left(\int_0^{1-e^{-t}} |K_1(x,t)|^\rho dx\right)^{1/\rho} \quad \le \quad C(1+t)(1-e^{-t})^{1/\rho} \quad \text{for all} \quad t\in(0,\infty)\,.$$

Proof. One can write

$$\left(\int_0^{1-e^{-t}} |K_1(x,t)|^\rho dx\right)^{1/\rho} \le C_M e^{t(1-1/\rho)}\left(\int_0^{e^t-1} \left((e^t+1)^2-y^2\right)^{-\frac{\rho}{2}}\Big|F\left(\frac{1}{2},\frac{1}{2};1;\frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2}\right)\Big|^\rho dy\right)^{1/\rho}.$$

Denote $z:=e^t>1$ and consider the integral $\displaystyle\int_0^{z-1}\left|\frac{1}{\sqrt{(1+z)^2-x^2}}F\Big(\frac{1}{2},\frac{1}{2};1;\frac{(z-1)^2-x^2}{(z+1)^2-x^2}\Big)\right|^\rho dx$ of the right-hand side. Then we apply Lemma 7.2 and obtain

$$\left(\int_0^{1-e^{-t}} |K_1(x,t)|^\rho dx\right)^{1/\rho} \le Ce^{t(1-1/\rho)}(1+\ln e^t)(e^t-1)^{1/\rho}(e^t+1)^{-1} \le C(1+t)(1-e^{-t})^{1/\rho}\,.$$

Proposition is proven. $\square$

Thus, the theorem in the case of $\varphi_0=0$ is proven.

Now we turn to the case of $\varphi_1=0$, where the kernel K_0 appears. The application of Theorem 0.4 leads to

$$\|u(x,t)\|_{L^q(\mathbb{R}_x)} \quad \le \quad e^{\frac{t}{2}}\|\varphi_0(x)\|_{L^q(\mathbb{R}_x)} + \left\|\int_0^{1-e^{-t}} \left[\varphi_0(x-z)+\varphi_0(x+z)\right]K_0(z,t)\,dz\right\|_{L^q(\mathbb{R}_x)}.$$

Similarly to the case of the second datum we arrive at

$$\|u(x,t)\|_{L^q(\mathbb{R}_x)} \quad \le \quad e^{\frac{t}{2}}\|\varphi_0(x)\|_{L^q(\mathbb{R}_x)} + \|\varphi_0(x)\|_{L^p(\mathbb{R}_x)}\left(\int_0^{1-e^{-t}} |K_0(r,t)|^\rho dr\right)^{1/\rho}.$$

The next proposition gives an estimate for the integral of the last inequality.

Proposition 8.3 *Let* $1<p<\rho'$, $\frac{1}{q}=\frac{1}{p}-\frac{1}{\rho'}$, $\frac{1}{\rho}+\frac{1}{\rho'}=1$, *and* $\rho\in[1,2)$. *We have*

$$\left(\int_0^{1-e^{-t}} |K_0(r,t)|^\rho dr\right)^{1/\rho} \quad \le \quad C_\rho(1+t)(e^t-1)^{\frac{1}{\rho}}e^{t(\frac{1}{2}-\frac{1}{\rho})} \quad \text{for all} \quad t\in(0,\infty)\,.$$

Proof. We turn to the integral ($z:=e^t>1$)

$$\begin{aligned}\left(\int_0^{1-e^{-t}} |K_0(r,t)|^\rho dr\right)^{1/\rho} = & \left(\int_0^{z-1} dy\left(\frac{1}{[(z-1)^2-y^2]\sqrt{(z+1)^2-y^2}}\right)^\rho\right.\\ & \times\Big|\big(z-z^2-iM(1-z^2-y^2)\big)F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(z-1)^2-y^2}{(z+1)^2-y^2}\Big)\\ & \left.+\big(z^2-1+y^2\big)\Big(\frac{1}{2}-iM\Big)F\Big(-\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(z-1)^2-y^2}{(z+1)^2-y^2}\Big)\Big|^\rho\right)^{1/\rho}.\end{aligned}$$

The formulas (7.7) and (7.8) describe the behavior of the hypergeometric functions in the neighbourhood of zero. Consider therefore two zones, $Z_1(\varepsilon,z)$ and $Z_2(\varepsilon,z)$, defined in (7.5) and (7.6), respectively. We split integral into two parts:

$$\int_0^{1-e^{-t}} |K_0(r,t)|^\rho\,dr \quad = \quad \int_{(z,r)\in Z_1(\varepsilon,z)} |K_0(r,t)|^\rho\,dr + \int_{(z,r)\in Z_2(\varepsilon,z)} |K_0(r,t)|^\rho\,dr\,.$$

In the proof of Lemma 7.4 the relation (7.9) was checked in the first zone. If $1 \le z \le N$ with some constant N, then the argument of the hypergeometric functions is bounded,

$$\frac{(z-1)^2-r^2}{(z+1)^2-r^2} \le \frac{(z-1)^2}{(z+1)^2} \le \frac{(N-1)^2}{(N+1)^2} < 1 \quad \text{for all} \quad r \in (0, z-1), \tag{8.2}$$

and we obtain with $z = e^t$,

$$\begin{aligned}
\left(\int_0^{1-e^{-t}} |K_0(r,t)|^\rho \, dr \right)^{1/\rho} &\le C_{M,N} \Bigg(\int_0^{z-1} \Bigg[\frac{1}{[(z-1)^2-y^2]\sqrt{(z+1)^2-y^2}} \\
&\quad \times \left\{ \frac{1}{2}[(z-1)^2-y^2] + z^2 \frac{(z-1)^2-y^2}{(z+1)^2-y^2} + z^2 \left(\frac{(z-1)^2-y^2}{(z+1)^2-y^2} \right)^2 \right\} \Bigg]^\rho dy \Bigg)^{1/\rho} \\
&\le C_{M,N} \left(\int_0^{z-1} \left[\frac{1}{\sqrt{(z+1)^2-y^2}} \left\{ 1 + z^2 \frac{1}{(z+1)^2-y^2} \right\} \right]^\rho dy \right)^{1/\rho} \\
&\le C_{M,N} (z-1)^{1/\rho} (z+1)^{-1} .
\end{aligned}$$

Thus, we can restrict ourselves to the case of large $z \ge N$ in both zones. Consider therefore for $\rho \in (1,2)$ the following integrals over the first zone

$$\begin{aligned}
A_4 &:= \int_{(z,r)\in Z_1(\varepsilon)} \left(\frac{1}{\sqrt{(z+1)^2-r^2}} \right)^\rho dr \le \int_0^{z-1} \left(\frac{1}{\sqrt{(z+1)^2-r^2}} \right)^\rho dr \\
&\le C(z-1)(z+1)^{-\rho} F\Big(\frac{1}{2}, \frac{\rho}{2}; \frac{3}{2}; \frac{(z-1)^2}{(z+1)^2}\Big) \\
&\le C(z-1)(z+1)^{-\rho} , \\
A_5 &:= \int_{(z,r)\in Z_1(\varepsilon)} \left(\frac{z^2}{[(z+1)^2-r^2]\sqrt{(z+1)^2-r^2}} \right)^\rho dr \le C z^{2\rho} (z-1)(z+1)^{-3\rho} F\Big(\frac{1}{2}, \frac{3\rho}{2}; \frac{3}{2}; \frac{(z-1)^2}{(z+1)^2}\Big).
\end{aligned}$$

Then, according to (15.3.6) of Ch.15[1] and [2],

$$\begin{aligned}
F(a,b;c;\zeta) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b;a+b-c+1;1-\zeta) \\
&\quad + (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a,c-b;c-a-b+1;1-\zeta)
\end{aligned} \tag{8.3}$$

for all $\zeta \in \mathbb{C}$, $|\arg(1-\zeta)| < \pi$. We use (8.3) with

$$\zeta = \frac{(z-1)^2}{(z+1)^2}, \qquad 1-\zeta = \frac{4z}{(z+1)^2}$$

to obtain for $\rho < 2$ and large $z \ge N$ the following estimate for the hypergeometric function,

$$F\Big(\frac{1}{2}, \frac{3\rho}{2}; \frac{3}{2}; \frac{(z-1)^2}{(z+1)^2}\Big) \le C(z+1)^{-1+\frac{3\rho}{2}} . \tag{8.4}$$

Thus,

$$A_5 \le C(z-1)(z+1)^{-1+\frac{\rho}{2}} .$$

For the next term we obtain a similar estimate,

$$A_6 := \int_{(z,r)\in Z_1(\varepsilon)} \left| \frac{z^2}{((z-1)^2-r^2)\sqrt{(z+1)^2-r^2}} \left(\frac{(z-1)^2-r^2}{(z+1)^2-r^2} \right)^2 \right|^\rho dr \le C(z-1)(z+1)^{-1+\frac{\rho}{2}} .$$

Hence,

$$\int_{(z,r)\in Z_1(\varepsilon,z)} |K_0(r,t)|^\rho \, dr \le C(z-1)(z+1)^{-1+\frac{\rho}{2}} \, .$$

In the second zone $Z_2(\varepsilon,z)$ for the argument of the hypergeometric functions we have (7.10), (7.13), and (7.14). We have to estimate the following integral

$$A_7 \quad := \quad \int_{(z,r)\in Z_2(\varepsilon,z)} \left| \frac{z^2\,(1+\ln z)}{((z-1)^2-r^2)\sqrt{(z+1)^2-r^2}} \right|^\rho dr \, .$$

We apply (7.10) and (8.4) to obtain

$$A_7 \quad \le \quad C\,(1+\ln z)^\rho \,(z-1)(z+1)^{-1+\frac{\rho}{2}} \, .$$

Hence,

$$\int_{(z,r)\in Z_2(\varepsilon,z)} |K_0(r,t)|^\rho \, dr \le C\,(1+\ln z)^\rho\,(z-1)(z+1)^{-1+\frac{\rho}{2}} \qquad \text{for all} \quad z \ge N \, .$$

Proposition is proven. $\square$

9 $L^p - L^q$ Estimates for the Equation with Source, $n \ge 2$

For the wave equation the Duhamel's principle allows to reduce the case of source term to the case of the Cauchy problem without source term and consequently to derive the $L^p - L^q$-decay estimates for the equation. For (0.9) the Duhamel's principle is not applicable straightforward and we have to appeal to the representation formula of Theorem 0.5. In fact, one can regard that formula as an expansion of the two-stage Duhamel's principle. In this section we consider the Cauchy problem (0.21) for the equation with the source term with zero initial data.

Theorem 9.1 *Let $u = u(x,t)$ be solution of the Cauchy problem (0.21). Then for $n > 1$ one has the following decay estimate*

$$\begin{aligned} & \|(-\triangle)^{-s}u(x,t)\|_{L^q(\mathbb{R}^n)} \\ \le \quad & C\int_0^t db\,\|f(x,b)\|_{L^p(\mathbb{R}^n)} \int_0^{e^{-b}-e^{-t}} dr\, r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(e^{-t}+e^{-b})^2-r^2}} F\left(\frac{1}{2},\frac{1}{2};1;\frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2}\right) \end{aligned}$$

provided that $s \ge 0$, $1 < p \le 2$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{2}(n+1)\left(\frac{1}{p}-\frac{1}{q}\right) \le 2s \le n\left(\frac{1}{p}-\frac{1}{q}\right) < 2s+1$.

Proof. In both cases, of even and odd n, one can write the representation (0.24). Due to the results of [5, 23] for the wave equation, we have

$$\begin{aligned} & \|(-\triangle)^{-s}u(x,t)\|_{L^q(\mathbb{R}^n)} \\ \le \quad & C\int_0^t db \int_0^{e^{-b}-e^{-t}} \|(-\triangle)^{-s}v(x,r;b)\|_{L^q(\mathbb{R}^n)} \frac{1}{\sqrt{(e^{-t}+e^{-b})^2-r^2}} F\left(\frac{1}{2},\frac{1}{2};1;\frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2}\right) dr \\ \le \quad & C\int_0^t db\,\|f(x,b)\|_{L^p(\mathbb{R}^n)} \int_0^{e^{-b}-e^{-t}} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(e^{-t}+e^{-b})^2-r^2}} F\left(\frac{1}{2},\frac{1}{2};1;\frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2}\right) dr \, . \end{aligned}$$

The theorem is proven. $\square$

We are going to transform the estimate of the last theorem to more cosy form. To this aim we estimate for $n(\frac{1}{p}-\frac{1}{q}) < 2s+1$ the last integral of the right hand side. If we replace $e^{-b}/e^{-t} > 1$ with $z := e^{-b}/e^{-t} > 1$, then the integral will be simplified.

$$
\begin{aligned}
& \int_0^{e^{-b}-e^{-t}} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(e^{-t}+e^{-b})^2-r^2}} F\left(\frac{1}{2},\frac{1}{2};1;\frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2}\right) dr \\
= \quad & e^{-t[2s-n(\frac{1}{p}-\frac{1}{q})]} \int_0^{z-1} y^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2-y^2}} F\left(\frac{1}{2},\frac{1}{2};1;\frac{(z-1)^2-y^2}{(z+1)^2-y^2}\right) dy
\end{aligned}
$$

Lemma 9.2 [33, Lemma 9.2] *Assume that* $0 \geq 2s-n(\frac{1}{p}-\frac{1}{q}) > -1$. *Then*

$$
\int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2-r^2}} F\left(\frac{1}{2},\frac{1}{2};1;\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\right) dr \leq Cz^{-1}(z-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})}(1+\ln z),
$$

for all $z > 1$.

Corollary 9.3 *Let* $u = u(x,t)$ *be solution of the Cauchy problem (0.21). Then for* $n \geq 2$ *one has the following decay estimate*

$$
\|(-\triangle)^{-s}u(x,t)\|_{L^q(\mathbb{R}^n)} \leq C\int_0^t \|f(x,b)\|_{L^p(\mathbb{R}^n)} e^{-b}\left(e^{-b}-e^{-t}\right)^{1+2s-n(\frac{1}{p}-\frac{1}{q})}(1+t-b)\, db \tag{9.1}
$$

provided that $s \geq 0$, $1 < p \leq 2$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{2}(n+1)\left(\frac{1}{p}-\frac{1}{q}\right) \leq 2s \leq n\left(\frac{1}{p}-\frac{1}{q}\right) < 2s+1$.

Proof. Indeed, we apply Lemma 9.2 with $z = e^{t-b}$ to the right-hand side of the estimate given by Theorem 9.1 :

$$
\begin{aligned}
\|(-\triangle)^{-s}u(x,t)\|_{L^q(\mathbb{R}^n)} & \leq C\int_0^t db\, \|f(x,b)\|_{L^p(\mathbb{R}^n)} e^{-t[2s-n(\frac{1}{p}-\frac{1}{q})]} z^{-1}(z-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})}(1+\ln z) \\
& \leq C\int_0^t \|f(x,b)\|_{L^p(\mathbb{R}^n)} e^{-b}\left(e^{-b}-e^{-t}\right)^{1+2s-n(\frac{1}{p}-\frac{1}{q})}(1+t-b)\, db\,.
\end{aligned}
$$

Corollary is proven. □

10 $L^p - L^q$ Estimates for Equation without Source, $n \geq 2$

The $L^p - L^q$-decay estimates for the energy of the solution of the Cauchy problem for the wave equation without source can be proved by the representation formula, $L_1 - L_\infty$ and $L_2 - L_2$ estimates, and interpolation argument. (See, e.g., [25, Theorem 2.1].) There is also a proof of the $L^p - L^q$-decay estimates that is based on the microlocal consideration and dyadic decomposition of the phase space. (See, e.g., [5, 23].) To avoid the derivative loss and obtain more sharp estimates we appeal to the representation formula provided by Theorem 0.6.

Theorem 10.1 *The solution* $u = u(x,t)$ *of the Cauchy problem (0.25) satisfies the following* $L^p - L^q$ *estimate*

$$
\|(-\triangle)^{-s}u(x,t)\|_{L^q(\mathbb{R}^n)} \leq C(1+t)(1-e^{-t})^{2s-n(\frac{1}{p}-\frac{1}{q})}\left\{e^{\frac{t}{2}}\|\varphi_0(x)\|_{L^p(\mathbb{R}^n)} + (1-e^{-t})\|\varphi_1\|_{L^p(\mathbb{R}^n)}\right\}
$$

for all $t \in (0,\infty)$, *provided that* $s \geq 0$, $1 < p \leq 2$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{2}(n+1)\left(\frac{1}{p}-\frac{1}{q}\right) \leq 2s \leq n\left(\frac{1}{p}-\frac{1}{q}\right) < 2s+1$.

Proof. We start with the case of $\varphi_0 = 0$. Due to Theorem 0.6 for the solution $u = u(x,t)$ of the Cauchy problem (0.25) with $\varphi_0 = 0$ and to the results of [5, 23] we have:

$$\begin{aligned}
&\|(-\triangle)^{-s} u(x,t)\|_{L^q(\mathbb{R}^n)} \\
\leq\ & C\|\varphi_1\|_{L^p(\mathbb{R}^n)} \int_0^{1-e^{-t}} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \, |K_1(r,t)| \, dr \\
\leq\ & C\|\varphi_1\|_{L^p(\mathbb{R}^n)} e^{-t[2s-n(\frac{1}{p}-\frac{1}{q})]} \int_0^{e^t-1} y^{2s-n(\frac{1}{p}-\frac{1}{q})} \big((e^t+1)^2 - y^2\big)^{-\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t-1)^2 - y^2}{(e^t+1)^2 - y^2}\right) dy \,.
\end{aligned}$$

To continue we apply Lemma 9.2 and obtain

$$\|(-\triangle)^{-s} u(x,t)\|_{L^q(\mathbb{R}^n)} \quad \leq \quad C\|\varphi_1\|_{L^p(\mathbb{R}^n)} (1+t)(1-e^{-t})^{1+2s-n(\frac{1}{p}-\frac{1}{q})} \,.$$

Thus, in the case of $\varphi_0 = 0$ the theorem is proven.

Next we turn to the case of $\varphi_1 = 0$. Due to Theorem 0.6 for the solution $u = u(x,t)$ of the Cauchy problem (0.25) with $\varphi_1 = 0$ and to the results of [5, 23] we have:

$$\|(-\triangle)^{-s} u(x,t)\|_{L^q(\mathbb{R}^n)} \quad \leq \quad C\left(e^{\frac{t}{2}} (1-e^{-t})^{2s-n(\frac{1}{p}-\frac{1}{q})} + \int_0^{1-e^{-t}} r^{2s-n(\frac{1}{p}-\frac{1}{q})} |K_0(r,t)| \, dr \right) \|\varphi_0(x)\|_{L^p(\mathbb{R}^n)}.$$

One can estimate the last integral

$$\begin{aligned}
& \int_0^{1-e^{-t}} r^{2s-n(\frac{1}{p}-\frac{1}{q})} |K_0(r,t)| \, dr \\
\leq\ & e^{-t[2s-n(\frac{1}{p}-\frac{1}{q})]} \int_0^{e^t-1} y^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{[(e^t-1)^2 - y^2]\sqrt{(e^t+1)^2 - y^2}} \\
& \times \Bigg| \big(e^t - e^{2t} - iM(1-e^{2t}-y^2)\big) F\Big(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(e^t-1)^2 - y^2}{(e^t+1)^2 - y^2}\Big) \\
& \qquad + \big(e^{2t} - 1 + y^2\big)\Big(\frac{1}{2} - iM\Big) F\Big(-\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(e^t-1)^2 - y^2}{(e^t+1)^2 - y^2}\Big) \Bigg| \, dy \,.
\end{aligned}$$

The following proposition gives the remaining estimate for that integral and completes the proof of the theorem.

Proposition 10.2 *If* $2s - n(\frac{1}{p} - \frac{1}{q}) > -1$, *then*

$$\begin{aligned}
& \int_0^{z-1} y^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{[(z-1)^2 - y^2]\sqrt{(z+1)^2 - y^2}} \\
& \times \Bigg| \big(z - z^2 - iM(1-z^2-y^2)\big) F\Big(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\Big) \\
& \qquad + \big(z^2 - 1 + y^2\big)\Big(\frac{1}{2} - iM\Big) F\Big(-\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\Big) \Bigg| \, dy \\
\leq\ & C z^{-\frac{1}{2}} (z-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})} (1 + \ln z) \qquad \textit{for all} \quad z > 1.
\end{aligned}$$

Proof. We follow the arguments have been used in the proof of Proposition 8.3. If $1 \leq z \leq N$ with some constant N, then the argument of the hypergeometric functions is bounded (8.2), and the integral can be estimated by:

$$\int_0^{z-1} y^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{[(z-1)^2 - y^2]\sqrt{(z+1)^2 - y^2}}$$

$$
\begin{aligned}
&\times\Big|\big(z-z^2-iM(1-z^2-y^2)\big)F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(z-1)^2-y^2}{(z+1)^2-y^2}\Big)\\
&\qquad +\big(z^2-1+y^2\big)\Big(\frac{1}{2}-iM\Big)F\Big(-\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(z-1)^2-y^2}{(z+1)^2-y^2}\Big)\Big|\,dy\\
\le\ & C_M\int_0^{z-1} y^{2s-n(\frac{1}{p}-\frac{1}{q})}\left[\frac{1}{\sqrt{(z+1)^2-y^2}}\left\{1+z^2\frac{1}{(z+1)^2-y^2}\right\}\right]dy\\
\le\ & C_M z^{-1}(z-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})}\quad\text{for all}\quad z\in[1,N]\,.
\end{aligned}
$$

Thus, we can restrict ourselves to the case of large $z\ge M$ in both zones $Z_1(\varepsilon,z)$ and $Z_2(\varepsilon,z)$, defined in (7.5) and (7.6), respectively. In the first zone we have (7.9). Consider therefore the following inequalities,

$$
\begin{aligned}
A_8 &:= \int_{(z,r)\in Z_1(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})}\frac{1}{\sqrt{(z+1)^2-r^2}}\,dr\\
&\le Cz^{-1}(z-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})}\quad\text{for all}\quad z\in[N,\infty)\,.
\end{aligned}
$$

For $0\ge a>-1$ and $z\ge N$ the following integral can be easily estimated:

$$
\begin{aligned}
\int_0^{z-1} r^a\frac{1}{((z+1)^2-r^2)^{3/2}}dr &= \int_0^{z/2} r^a\frac{1}{((z+1)^2-r^2)^{3/2}}dr+\int_{z/2}^{z-1} r^a\frac{1}{((z+1)^2-r^2)^{3/2}}dr\\
&\le \frac{16}{9}z^{-3}\int_0^{z/2} r^a dr+\frac{z^a}{4^a}\int_{z/2}^{z-1}\frac{1}{((z+1)^2-r^2)^{3/2}}dr\\
&\le Cz^{a-3/2}\quad\text{for all}\quad z\in[N,\infty)\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
A_9 &:= z^2\int_{(z,r)\in Z_1(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})}\frac{1}{\sqrt{(z+1)^2-r^2}}\frac{1}{(z+1)^2-r^2}dr\\
&\le z^2\int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})}\frac{1}{\sqrt{(z+1)^2-r^2}}\frac{1}{(z+1)^2-r^2}dr\\
&\le Cz^{-\frac{1}{2}}(z-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})}\quad\text{for all}\quad z\in[N,\infty)\,,
\end{aligned}
$$

and

$$
\begin{aligned}
A_{10} &:= z^2\int_{(z,r)\in Z_1(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})}\frac{1}{((z-1)^2-r^2)\sqrt{(z+1)^2-r^2}}\left(\frac{(z-1)^2-r^2}{(z+1)^2-r^2}\right)^2dr\\
&\le z^2\int_{(z,r)\in Z_1(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})}\frac{1}{\sqrt{(z+1)^2-r^2}}\frac{1}{(z+1)^2-r^2}dr\\
&\le Cz^{-\frac{1}{2}}(z-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})}\quad\text{for all}\quad z\in[N,\infty)\,.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\int_{(z,y)\in Z_1(\varepsilon,z)} y^{2s-n(\frac{1}{p}-\frac{1}{q})}\frac{1}{[(z-1)^2-y^2]\sqrt{(z+1)^2-y^2}}\\
&\times\Big|\big(z-z^2-iM(1-z^2-y^2)\big)F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(z-1)^2-y^2}{(z+1)^2-y^2}\Big)\\
&\qquad +\big(z^2-1+y^2\big)\Big(\frac{1}{2}-iM\Big)F\Big(-\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(z-1)^2-y^2}{(z+1)^2-y^2}\Big)\Big|\,dy\\
\le\ & Cz^{-\frac{1}{2}}(z-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})}\quad\text{for all}\quad z\in[1,\infty)\,.
\end{aligned}
$$

In the second zone we use (7.10), (7.11), and (7.14). Thus, we have to estimate the next two integrals:

$$\begin{aligned}
A_{11} &:= z^2 \int_{(z,r)\in Z_2(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{((z-1)^2-r^2)\sqrt{(z+1)^2-r^2}}\, dr\,, \\
A_{12} &:= z^2\,(1+\ln z) \int_{(z,r)\in Z_2(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{((z-1)^2-r^2)\sqrt{(z+1)^2-r^2}}\, dr\,.
\end{aligned}$$

We apply (7.10) to A_{11} and obtain

$$A_{11} \le C_\varepsilon z^2 \int_{(z,r)\in Z_2(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{[(z+1)^2-r^2]} \frac{1}{\sqrt{(z+1)^2-r^2}}\, dr \le C_\varepsilon z^{-\frac{1}{2}}(z-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})}$$

for all $z\in[1,\infty)$, while

$$A_{12} \quad \le \quad C_\varepsilon z^{-\frac{1}{2}}(z-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})}\,(1+\ln z) \quad \text{for all} \quad z\in[1,\infty)\,.$$

Proposition is proven. $\square$

To complete the proof of the theorem we write

$$\begin{aligned}
& \int_0^{1-e^{-t}} r^{2s-n(\frac{1}{p}-\frac{1}{q})}|K_0(r,t)|\,dr \\
\le\ & e^{-t[2s-n(\frac{1}{p}-\frac{1}{q})]}\int_0^{e^t-1} y^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{[(e^t-1)^2-y^2]\sqrt{(e^t+1)^2-y^2}} \\
& \times\Big|\big(e^t-e^{2t}-iM(1-e^{2t}-y^2)\big)F\Big(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2}\Big) \\
& \qquad +\big(e^{2t}-1+y^2\big)\Big(\frac{1}{2}-iM\Big)F\Big(-\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2}\Big)\Big|\,dy \\
\le\ & Ce^{-t[\frac{1}{2}+2s-n(\frac{1}{p}-\frac{1}{q})]}(e^t-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})}(1+t)\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \|(-\triangle)^{-s}u(x,t)\|_{L^q(\mathbb{R}^n)} \\
\le\ & C\Big(e^{\frac{t}{2}}(1-e^{-t})^{2s-n(\frac{1}{p}-\frac{1}{q})}+e^{-t[\frac{1}{2}+2s-n(\frac{1}{p}-\frac{1}{q})]}(e^t-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})}(1+t)\Big)\|\varphi_0(x)\|_{L^p(\mathbb{R}^n)} \\
\le\ & C(1+t)e^{\frac{t}{2}}(1-e^{-t})^{2s-n(\frac{1}{p}-\frac{1}{q})}\|\varphi_0(x)\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Theorem is proven. $\square$

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